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Number theory problems from the harmonic analysis of a fractal

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We study some number theory problems related to the harmonic analysis (Fourier bases) of the Cantor set introduced by Jorgensen and Pedersen in [JP98].

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1. Introduction

In [JP98], Jorgensen and Pedersen made a surprising discovery: they constructed a fractal measure on a Cantor set which has an orthonormal Fourier series. This Cantor set is obtained from the interval $[0, 1]$, dividing it into four equal intervals and keeping the first and the third, $[0, 1/4]$ and $[1/2, 3/4]$, and repeating the procedure infinitely many times. It can be described in terms of iterated function systems: let

$$\tau_0(x) = x/4 \text{ and } \tau_2(x) = (x + 2)/4, \quad (x \in \mathbb{R}).$$

The Cantor set X_4 is the unique compact set that satisfies the invariance condition

$$X_4 = \tau_0(X_4) \cup \tau_2(X_4).$$

The set X_4 is described also in terms of the base 4 decomposition of real numbers:

$$X_4 = \left\{ \sum_{k=1}^{n} 4^{-k} b_k : b_k \in \{0, 2\}, n \in \mathbb{N} \right\}.$$

On the set X_4 one considers the Hausdorff measure μ of dimension $\log_4 2 = \frac{1}{2}$. In terms of iterated function systems, the measure μ is the invariant measure for the iterated function system, that is, the unique Borel probability measure that satisfies the invariance equation

$$\mu(E) = \frac{1}{2} \left(\mu(\tau_0^{-1} E) + \mu(\tau_2^{-1} E) \right), \text{ for all Borel sets } E \subset \mathbb{R}. \tag{1.1}$$

Equivalently, for all continuous compactly supported functions f,

$$\int f \, d\mu = \frac{1}{2} \left(\int f \circ \tau_0 \, d\mu + \int f \circ \tau_2 \, d\mu \right). \tag{1.2}$$

We denote, for $\lambda \in \mathbb{R}$:

$$e_\lambda(x) = e^{2\pi i \lambda \cdot x}, \quad (x \in \mathbb{R}).$$

Jorgensen and Pedersen proved that the Hilbert space $L^2(\mu)$ has an orthonormal basis formed with exponential functions, i.e., a Fourier basis, $E(\Gamma_0) := \{e_\lambda : \lambda \in \Gamma_0\}$ where

$$\Gamma_0 := \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0, 1\}, n \in \mathbb{N} \right\}. \tag{1.3}$$