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Positive density of integer polynomials with some prescribed properties

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ABSTRACT

In this paper, we show that various kinds of integer polynomials with prescribed properties of their roots have positive density. For example, we prove that almost all integer polynomials have exactly one or two roots with maximal modulus. We also show that for any positive integer n and any set of n distinct points symmetric with respect to the real line, there is a positive density of integer polynomials of degree n, height at most H and Galois group S_n whose roots are close to the given n points.

1. Introduction

The so-called *fundamental theorem of algebra*, first proved by Gauss, asserts that every non-constant univariate polynomial with complex coefficients has at least one complex

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root. Polynomials and various properties of their roots often play a crucial role in studying many mathematical objects.

For example, the properties of linear recurrence sequences (LRS) rely heavily on their characteristic polynomials. Recall that a LRS $\{u_n\}_{n=0}^{\infty}$ of order $n \ge 2$ with elements in $\mathbb{C}$ is defined by the linear relation

$$u_{m+n} = a_1 u_{m+n-1} + \cdots + a_n u_m \quad (m = 0, 1, 2, \dots),$$

where $a_1, \dots, a_n \in \mathbb{C}$ (called the *coefficients*), $a_n \ne 0$ and $u_j \ne 0$ for at least one j in the range $0 \le j \le n-1$. The characteristic polynomial of this LRS is

$$f(X) = X^n - a_1 X^{n-1} - \cdots - a_n = \prod_{i=1}^{n}(X - \alpha_i) \in \mathbb{C}[X],$$

where all α_i are said to be the *characteristic roots* of the sequence $\{u_n\}_{n=0}^{\infty}$. The sequence $\{u_n\}_{n=0}^{\infty}$ is called *non-degenerate* if the quotient of any of its two distinct characteristic roots is not a root of unity. Then, there are only finitely many integers n with $u_n = 0$ if the sequence is non-degenerate; otherwise, there may be infinitely many such integers. It has been proved in [11] that almost all LRS with integer (or rational) coefficients are non-degenerate.

In this paper, a polynomial is called *dominant* if it has a simple root (called *dominant root*) whose modulus is strictly greater than the moduli of its remaining roots. The LRS with dominant characteristic polynomial (called *dominant LRS*) are often much easier to deal with, especially when considering Diophantine properties of linear recurrence sequences.

In order to interpret the meanings of "almost all" and "positive density", we introduce the following definition, which only shows the right way to comprehend "almost all" and "positive density", but does not explain all their meanings in this paper. Given a proposition **P** related to integer polynomials, for integers $n \ge 1$ and $H \ge 1$, we define the set

$$\mathscr{P}_n(H) = \{f(X) = X^n + a_1 X^{n-1} + \cdots + a_n \in \mathbb{Z}[X] : \\ |a_i| \le H,\ i = 1, \dots, n\}.$$

We say that **P** is true *for almost all* (or *with density tending to* 1) monic integer polynomials if for any integer $n \ge 1$ we have

$$\lim_{H \to \infty} \frac{|\{f \in \mathscr{P}_n(H) : \mathbf{P} \text{ is true for } f\}|}{|\mathscr{P}_n(H)|} = 1.$$

Throughout, we use $|T|$ to denote the cardinality of a finite set T. In addition, we say that **P** is true *with positive density* if for any integer $n \ge 1$ the following holds:

$$\limsup_{H \to \infty} \frac{|\{f \in \mathscr{P}_n(H) : \mathbf{P} \text{ is true for } f\}|}{|\mathscr{P}_n(H)|} > 0,$$