Journal of Number Theory 159 (2016) 45–58

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Sum of one prime and two squares of primes in short intervals

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A R T I C L E   I N F O

Article history:
Received 13 October 2014
Received in revised form 29 July 2015
Accepted 29 July 2015
Available online 2 September 2015
Communicated by David Goss

MSC:
primary 11P32
secondary 11P55, 11P05

Keywords:
Waring–Goldbach problem
Laplace transforms

A B S T R A C T

Assuming the Riemann Hypothesis we prove that the interval $[N, N+H]$ contains an integer which is a sum of a prime and two squares of primes provided that $H \geq C(\log N)^4$, where $C > 0$ is an effective constant.

1. Introduction

The problem of representing an integer as a sum of a prime and of two prime squares is classical. Letting

$$\mathcal{A} = \{n \in \mathbb{N} : n \equiv 1 \bmod 2;\ n \not\equiv 2 \bmod 3\},$$

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http://dx.doi.org/10.1016/j.jnt.2015.07.010

it is conjectured that every sufficiently large $n \in \mathcal{A}$ can be represented as $n = p_1 + p_2^2 + p_3^2$. Let now N be a large integer. Several results about the cardinality $E(N)$ of the set of integers $n \le N$, $n \in \mathcal{A}$ which are not representable as a sum of a prime and two prime squares were proved during the last 75 years; we recall the papers of Hua [3], Schwarz [17], Leung–Liu [11], Wang [18], Wang–Meng [19], Li [12] and Harman–Kumchev [2]. Recently L. Zhao [20] proved that

$$E(N) \ll N^{1/3+\varepsilon}.$$

As a consequence we can say that every integer $n \in [1, N] \cap \mathcal{A}$, with at most $\mathcal{O}\big(N^{1/3+\varepsilon}\big)$ exceptions, is the sum of a prime and two prime squares. Letting

$$r(n) = \sum_{p_1+p_2^2+p_3^2=n} \log p_1 \log p_2 \log p_3, \tag{1}$$

in fact L. Zhao also proved that a suitable asymptotic formula for $r(n)$ holds for every $n \in [1, N] \cap \mathcal{A}$, with at most $\mathcal{O}\big(N^{1/3+\varepsilon}\big)$ exceptions.

In this paper we study the average behaviour of $r(n)$ over short intervals $[N, N + H]$, $H = o(N)$. Assuming that the Riemann Hypothesis (RH) holds, we prove that a suitable asymptotic formula for such an average of $r(n)$ holds in short intervals with no exceptions.

Theorem 1. *Assume the Riemann Hypothesis. We have*

$$\sum_{n=N+1}^{N+H} r(n) = \frac{\pi}{4} HN + \mathcal{O}\Big(H^{1/2} N (\log N)^2 + HN^{3/4} (\log N)^3 + H^2 (\log N)^{3/2}\Big) \quad \text{as } N \to \infty,$$

uniformly for $\infty((\log N)^4) \le H \le o\big(N(\log N)^{-3/2}\big)$, *where* $f = \infty(g)$ *means* $g = o(f)$.

Letting

$$r^*(n) = \sum_{p_1+p_2^2+p_3^2=n} 1,$$

a similar asymptotic formula holds for the average of $r^*(n)$ too.

In the unconditional case our proof yields a weaker result than Zhao's, namely, the asymptotic formula for the average of $r(n)$ holds just for $H \ge N^{7/12+\varepsilon}$; for this reason, here we are only concerned with the conditional one. It is worth remarking that, under the assumption of RH, the formula in Theorem 1 implies that every interval $[N, N + H]$ contains an integer which is a sum of a prime and two prime squares, where $CL^4 \le H = o\big(NL^{-3/2}\big)$, $C > 0$ is a suitable large constant and $L = \log N$. We recall that the analogous results for the binary Goldbach problem are respectively $H \gg N^{c+\varepsilon}$ with $c = 21/800$, by Baker–Harman–Pintz and Jia, see [15], and $H \gg L^2$, under the assumption of RH; see, *e.g.*, [5]. Assuming RH, the expectation in Theorem 1 is the lower