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Ramanujan's Eisenstein series of level 7 and 14

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In this paper, we give an elementary proof of Ramanujan's Eisenstein series of level 7. In the process, we also prove four Eisenstein series of level 14 due to S. Cooper and D. Ye [4].

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1. Introduction

Let τ be a complex number satisfying $\operatorname{Im}(\tau) > 0$, and let $q = e^{2\pi i\tau}$. The Dedekind eta-function is defined by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Let $P(q)$ denote Ramanujan's Eisenstein series of weight 2, defined by

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}.$$

For any positive integer n, let P_n be defined by

$$P_n := P(q^n).$$

S. Ramanujan [7, p. 254] stated without proof the results which are equivalent to

$$-P_1 + 7P_7 = 6\left\{1 + \sum_{k=1}^{\infty} \left(\frac{k}{7}\right) \frac{q^k}{1 - q^k}\right\}^2 \tag{1.1}$$

$$= 6\left\{\frac{f_1^8 + 13qf_1^4f_7^4 + 49q^2f_7^8}{f_1f_7}\right\}^{2/3}. \tag{1.2}$$

$$-P_2 + 7P_{14} = 6\left\{\frac{f_2^5f_{14}^5}{f_1^2f_4^2f_7^2f_{28}^2} - 2q\frac{f_1f_4f_7f_{28}}{f_2f_{14}}\right\}^2 \tag{1.3}$$

$$= 3\left\{\frac{f_2^{10}f_{14}^{10}}{f_1^4f_4^4f_7^4f_{28}^4}\right\}\left\{1 + 16q^4\frac{f_1^4f_4^8f_7^4f_{28}^8}{f_2^{12}f_{14}^{12}} + \frac{f_1^8f_4^4f_7^8f_{28}^4}{f_2^{12}f_{14}^{12}}\right\}, \tag{1.4}$$

where $\left(\dfrac{k}{p}\right)$ is the Legendre symbol and $f_n = f(-q^n) = q^{-n/24}\eta(n\tau)$.

B.C. Berndt [3, pp. 467–473] found proofs of (1.1)–(1.4) by constructing certain differential equations satisfied by the quotients of eta-functions and some Ramanujan's modular equations of the seventh degree. Z.-G. Liu [5,6] proved (1.1) and (1.2) by using complex variable theory of elliptic functions and Ramanujan's modular equations of the seventh degree.

In this paper, we give a quite different proof of (1.1)–(1.4). Our approach is based on the identity (2.3) stated below, which is a particular case of Bailey's ${}_6\psi_6$ summation formula. In the process, we also obtain new proofs of four Eisenstein series of level 14 of S. Cooper and D. Ye [4]. In Section 2 of this paper, we recall some definitions, notations and certain theta-function identities of seventh degree, which will be used to prove the main results. In Section 3, we prove four Eisenstein series of level 14 of Cooper and Ye [4]. In Section 4, we establish (1.1)–(1.4).