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Two closed forms for the Bernoulli polynomials

Feng Qi [a,b,c,*], Robin J. Chapman [d]

[a] *Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China*
[b] *College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China*
[c] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China*
[d] *Mathematics Research Institute, University of Exeter, United Kingdom*

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In the paper, the authors find two closed forms involving the Stirling numbers of the second kind and in terms of a determinant of combinatorial numbers for the Bernoulli polynomials and numbers.

* Corresponding author.
E-mail addresses: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com (F. Qi), R.J.Chapman@exeter.ac.uk, rjc@maths.ex.ac.uk (R.J. Chapman).
URLs: https://qifeng618.wordpress.com (F. Qi), http://empslocal.ex.ac.uk/people/staff/rjchapma/rjc.html (R.J. Chapman).

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1. Introduction

It is common knowledge that the Bernoulli numbers and polynomials B_k and $B_k(u)$ for $k \ge 0$ satisfy $B_k(0) = B_k$ and can be generated respectively by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi$$

and

$$\frac{ze^{uz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(u) \frac{z^k}{k!}, \quad |z| < 2\pi.$$

Because the function $\frac{x}{e^x-1} - 1 + \frac{x}{2}$ is odd in $x \in \mathbb{R}$, all of the Bernoulli numbers B_{2k+1} for $k \in \mathbb{N}$ equal 0. It is clear that $B_0 = 1$ and $B_1 = -\frac{1}{2}$. The first few Bernoulli numbers B_{2k} are

$$\begin{aligned} B_2 &= \frac{1}{6}, & B_4 &= -\frac{1}{30}, & B_6 &= \frac{1}{42}, & B_8 &= -\frac{1}{30}, \\ B_{10} &= \frac{5}{66}, & B_{12} &= -\frac{691}{2730}, & B_{14} &= \frac{7}{6}, & B_{16} &= -\frac{3617}{510}. \end{aligned}$$

The first five Bernoulli polynomials are

$$\begin{gathered} B_0(u) = 1, \quad B_1(u) = u - \frac{1}{2}, \quad B_2(u) = u^2 - u + \frac{1}{6}, \\ B_3(u) = u^3 - \frac{3}{2}u^2 + \frac{1}{2}u, \quad B_4(u) = u^4 - 2u^3 + u^2 - \frac{1}{30}. \end{gathered}$$

In combinatorics, the Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 1$ can be computed and generated by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=1}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n \quad \text{and} \quad \frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}$$

respectively. See [7, p. 206].

It is easy to see that the generating function of $B_k(u)$ can be reformulated as

$$\frac{ze^{uz}}{e^z - 1} = \left[\frac{e^{(1-u)z} - e^{-uz}}{z}\right]^{-1} = \frac{1}{\int_{-u}^{1-u} e^{zt} \,\mathrm{d}\, t} = \frac{1}{\int_0^1 e^{z(t-u)} \,\mathrm{d}\, t}. \tag{1.1}$$

This expression will play important role in this paper. For related information on the integral expression (1.1), please refer to [12–14,31,32] and plenty of references cited in the survey and expository article [30].