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Arithmetic properties of partitions with designated summands

Ernest X.W. Xia

Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, PR China

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ABSTRACT

A new class of partitions, partitions with designated summands, was introduced by Andrews, Lewis and Lovejoy. Let $PD(n)$ denote the number of partitions of n with designated summands. Andrews, Lewis and Lovejoy established many congruences modulo 3 and powers of 2 for $PD(n)$ by using the theory of modular forms. In this paper, we prove several infinite families of congruences modulo 9 and 27 for $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+1)$ which were discovered by Chen, Ji, Jin and Shen. For example, we prove that for $n \geq 0$ and $k \geq 1$, $PD(2^{18k-1}(12n+1)) \equiv 0 \pmod{27}$. Furthermore, using some results due to Newman, we find some strange congruences modulo 27 for $PD(n)$. For example, we prove that for $k \geq 0$, $PD(13^{9k}(75p+2)) \equiv 0 \pmod{27}$ and $PD(2 \times 13^{9k+8}) \equiv 0 \pmod{27}$, where p is a prime and $p \equiv 1 \pmod{12}$.

1. Introduction

Andrews, Lewis and Lovejoy [1] investigated the number of partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one

E-mail address: Ernestxwxia@163.com.

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part among parts with equal size. For example, there are 15 partitions of 5 with designated summands:

$$5', \quad 4'+1', \quad 3'+2', \quad 3'+1'+1, \quad 3'+1+1', \quad 2'+2+1', \quad 2+2'+1',$$

$$2'+1'+1+1, \quad 2'+1+1'+1, \quad 2'+1+1+1', \quad 1'+1+1+1+1,$$

$$1+1'+1+1+1, \quad 1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

Let $PD(n)$ denote the number of partitions of n with designated summands. Thus, $PD(5) = 15$. As usual, set $PD(0) = 1$. Andrews, Lewis and Lovejoy [1] derived the following generating function of $PD(n)$:

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3} \tag{1.1}$$

where here and throughout this paper, for any positive integer k, f_k is defined by

$$f_k := \prod_{n=1}^{\infty}(1-q^{kn}). \tag{1.2}$$

The concept of partitions with designated summands goes back to MacMahon [9]. He considered partitions with designated summands wherein exactly k different magnitudes occur among all the parts, see also Andrews and Rose [2].

Andrews, Lewis and Lovejoy [1] obtained explicit formulas for the generating functions for $PD(2n)$ and $PD(2n+1)$. Moreover, they also proved many congruences modulo 3 and powers of 2 for $PD(n)$. In particular, they proved that for $n \geq 0$,

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{1.3}$$

Recently, Chen, Ji, Jin and Shen [6] obtained a Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the congruence of Andrews, Lewis and Lovejoy. They established explicit formulas for $PD(3n)$ and $PD(3n+1)$ by using some identities on Ramanujan's cubic continued fraction and cubic theta functions. Furthermore, they also gave a combinatorial interpretation of the congruence of Andrews, Lewis and Lovejoy for $PD(3n+2)$ by introducing a rank for partitions with designated summands.

In this paper, we prove several infinite families congruences modulo 9 and 27 for $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+2)$ due to Chen, Ji, Jin and Shen [6]. Moreover, we also establish some strange congruences modulo 27 for $PD(n)$ by utilizing some results given by Newman [10].

The main results of this paper can be stated as follows.