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On the distribution of squarefree numbers

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ABSTRACT

We develop estimates for multiple exponential sums, by which we can improve the result of 1993 on the distribution of squarefree numbers (under RH, as usual).

1. Introduction

Let $Q(x)$ be the number of squarefree (2-free) numbers not exceeding x, and x is a sufficiently large positive number. It is elementary to prove that

$$Q(x) = \sum_{n \le x} \left|\mu(n)\right| = 6\pi^{-2}x + \Delta(x),$$

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where $\Delta(x)$ is the error term, and $\Delta(x) = O(x^{1/2})$. The problem of estimating $\Delta(x)$ by assuming RH (the Riemann hypothesis of the zeta-function) was studied by Montgomery and Vaughan [MV] using exponential sums, and their result is

$$\Delta(x) = O\big(x^{9/28+\varepsilon}\big),$$

where ε is a sufficiently small given positive number. Graham [G] improved their result to

$$\Delta(x) = O\big(x^{8/25+\varepsilon}\big).$$

Baker and Pintz [BPi] utilized the method of Heath-Brown [HB] to get

$$\Delta(x) = O\big(x^{7/22+\varepsilon}\big).$$

Jia [J] presented a difficult new estimate for double exponential sums, which enabled him to get the improvement

$$\Delta(x) = O\big(x^{17/54+\varepsilon}\big).$$

Although this important result seems to be the limitation of the current methods of exponential sum methods, a reader familiar with van der Corput's method would still ask the question as whether we can get a new improvement. Recently [BPo] got hitherto best result for the distribution of k-free numbers for $3 \le k \le 5$, and [L] got some improvements on results of [BPo] and [BPi] for $k \ge 5$, but their methods cannot yield an improvement of Jia's result of [J] for squarefree (2-free) numbers. The purpose of our paper is to improve Jia's result. We are able to present new estimates for multiple exponential sums, from which we get a new result as follows (we only use the decomposition of the Möbius function of [J]).

Theorem 1. *Assuming RH, there holds*

$$\Delta(x) = O\big(x^{11/35+\varepsilon}\big).$$

It is notable that we shall derive a result for counting lattice points (see Lemma 6) by using a similar method of proving Lemma 1.1 of [L1], which will play an important role in our arguments. Our Lemma 7 may constitute the novel aspect in the theory of exponential sums, for in its proof we are unable to deal directly with the complicated functions on the exponent.

Notations. $e(\xi) = exp(2\pi i\xi)$ for a real number ξ. For two positive numbers t and T, $t \sim T$ means $1 < t/T \le 2$, and $t \approx T$ means that $C \le t/T \le C'$ for suitable constants $C > 0$, $C' > 0$. C_i $(i \ge 1)$ denotes a suitable constant. $[\xi]$ is the largest integer not exceeding ξ, $\|\xi\| = \min(\xi - [\xi], 1 - \xi + [\xi])$, and $\psi(\xi) = \xi - [\xi] - \frac{1}{2}$.