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Generalized iterations and primitive divisors

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ABSTRACT

Let $(g_i)_{i\geq 1}$ be a sequence of Chebyshev polynomials, each with degree at least two, and define $(f_i)_{i\geq 1}$ by the following recursion: $f_1 = g_1$, $f_n = g_n \circ f_{n-1}$, for $n \geq 2$. Choose $\alpha \in \mathbb{Q}$ such that $\{g_1^n(\alpha) : n \geq 1\}$ is an infinite set. The main result is as follows: If $f_n(\alpha) = \frac{A_n}{B_n}$ is written in lowest terms, then for all but finitely many $n > 0$, the numerator, A_n, has a primitive divisor; that is, there is a prime p which divides A_n but does not divide A_i for any $i < n$.

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1. Introduction

Let X be a set and ϕ a map from X to X. A dynamical system is the set X together with the function ϕ. In the study of arithmetic dynamics one studies arithmetic properties of a dynamical system. Let n be a positive integer, denote by ϕ^n the n-fold composition of ϕ with itself. That is

$$\phi^n := \underbrace{\phi \circ \phi \circ \ldots \circ \phi}_{n},$$

and ϕ^0 is simply the identity map. We define the forward orbit of a point α under ϕ to be the set $\{\alpha, \phi(\alpha), \phi^2(\alpha), \ldots\}$.

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Given a sequence of integers $(A_n)_{n\geq 1}$, a *primitive divisor* of an element A_n is a prime p such that $p|A_n$ yet $p \nmid A_i$ for all $1 \leq i < n$.

Arithmetic dynamics has been traditionally concerned with iteration of a single function. In [2], Ingram and Silverman proved a Bang–Zsigmondy result for iterations of a single map. This note will introduce a new extension of arithmetic dynamics, arithmetic of generalized iterations, and conclude by proving Bang–Zsigmondy result in the case of generalized iteration of a special family of maps, Chebyshev polynomials.

Generalized iteration is an extension of iteration of a single function. Let g_i be an indexed family of rational maps; we do not disallow $g_i = g_j$ for some $i \neq j$. As in [1], a *generalized iteration* is an iteration of the form $g_k(g_{k-1}(\ldots(g_1(x))))$. Iteration of a single function is a special case of generalized iteration, many of the concepts present in iteration of a single function can be generalized. Following [2], given a sequence of integers $(A_n)_{n\geq 1}$, we define the *Zsigmondy set of* $(A_n)_{n\geq 1}$ to be the set

$$\mathcal{Z}((A_n)_{n\geq 1}) = \{n \geq 1 : A_n \text{ does not have a primitive divisor}\}.$$

In this note we prove the following theorem.

Theorem 1.1 *(Main theorem). Let $\mathbf{f} = (f_i)_{i=1}^{\infty}$ be a generalized iteration of Chebyshev polynomials, and $\alpha \in \mathbb{Q}$ a wandering point (see Definition 2.5). If $f_n(\alpha) = \frac{A_n}{B_n}$ is written in lowest terms, then the dynamical Zsigmondy set $\mathcal{Z}((A_n)_{n\geq 1})$ is finite.*

The main focus on this paper will be the development of tools similar to those used by Ingram and Silverman [2] for the case of generalized iteration. These tools provide a powerful method of proving the main theorem and it is the author's hope that these tools may be useful for generalizing the main theorem to other families of commuting functions. However, it should be stated that a shorter proof of the main theorem does exist for Chebyshev polynomials over number fields, a sketch of the proof is included.[1]

Theorem 1.2 *(Main theorem over number fields). Let K be a number field and let $\alpha \in K$ be a wondering point. Let $\{k_i\}_{i\geq 1}$ be a sequence of integers with $k_i \geq 2$ for all $i \geq 1$. We set $f_n = T_{k_n} \circ \cdots \circ T_{k_1}$ and write the fractional ideal*

$$(f_n(\alpha)) = \mathfrak{A}_n \mathfrak{B}_n^{-1}$$

as a quotient of relatively prime integral ideals. Then the Zsigmondy set $\mathcal{Z}((\mathfrak{A}_n)_{n\geq 1})$ is finite.

The aforementioned theorem can be proven as a consequence of a result by Postnikova and Schinzel which states that given two relatively prime integers of a number field K for

[1] A more complete proof is covered in the author's thesis [8]. The author would also like to thank the anonymous referee for providing a more general proof independent of the author's thesis. The referee's more general proof will be outlined in this document.

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