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Some basic results in elementary number theory in function fields

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ABSTRACT

In this paper, we introduce a Carlitz module analogue of the classical Euler totient function, and prove a function field analogue of Euler's theorem by using the Carlitz action and the Carlitz module analogue of the Euler totient function. We propose a function field analogue of Carmichael's totient function conjecture. In contrast to the classical case, we answer the function field analogue of Carmichael's conjecture in the negative. We also propose a function field analogue of Sierpiński's conjecture, and discuss some special cases of this analogue.

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1. Introduction

Let q be a power of a prime p. Let $\mathbf{A} = \mathbb{F}_q[T]$, and let $\mathbf{k} = \mathbb{F}_q(T)$, where $\mathbb{F}_q$ is the finite field of q elements. Here and elsewhere, we also let $\mathbf{A}_+$ be the subset of $\mathbf{A}$ consisting of all monic polynomials in $\mathbf{A}$.

Let τ be the mapping defined by $\tau(x) = x^q$, and let $\mathbf{k}\langle\tau\rangle$ denote the twisted polynomial ring. Let $C : \mathbf{A} \to \mathbf{k}\langle\tau\rangle$ $(a \mapsto C_a)$ be the Carlitz module, namely, C is an $\mathbb{F}_q$-algebra homomorphism such that $C_T = T + \tau$. See Goss [7], Rosen [11], or Thakur [12] for more information about the Carlitz module.

In this note, we are concerned with studying a Carlitz module analogue of the classical Euler totient function, denoted by ϕ_C, that is defined on $\mathbf{A} \setminus \{0\}$, and takes values in $\mathbf{A}_+$. There is another function field analogue, say $\phi_{\mathbf{A}}$, of the classical Euler totient function whose definition can be found, for example, in Rosen [11, Proposition 1.7]. There are many distinct features between the function ϕ_C in this note and the function $\phi_{\mathbf{A}}$ in Rosen [11]. First, the function ϕ_C takes values in $\mathbf{A}_+$ whereas the function $\phi_{\mathbf{A}}$ takes values in $\mathbb{Z}$. Second, for each $m \in \mathbf{A} \setminus \{0\}$, the value of ϕ_C at m is associated to the additive group $\mathbf{A}/m\mathbf{A}$ whereas the value of $\phi_{\mathbf{A}}$ at m is associated to the multiplicative group $(\mathbf{A}/m\mathbf{A})^\times$. (In fact, $\phi_{\mathbf{A}}(m)$ is the number of elements in the group $(\mathbf{A}/m\mathbf{A})^\times$.)

Let us now describe the content of this paper. In Section 2, we introduce the analogue ϕ_C of the classical Euler totient function, and prove a Carlitz module analogue of Euler's theorem using the function ϕ_C and the Carlitz action (see Theorem 2.7). The latter is motivated by the well-known analogies between the Carlitz module $u \mapsto C_m(u)$, $m \in \mathbf{A}$ and the power map $u \mapsto u^m$, $m \in \mathbb{Z}$. Note that there is another function field analogue of Euler's theorem (see Rosen [11, Proposition 1.8]) that is based on the analogy between the groups $(\mathbf{A}/m\mathbf{A})^\times$ and $(\mathbb{Z}/m\mathbb{Z})^\times$. In the same section, we also prove some results that show the similarity between the function ϕ_C and the classical Euler totient function.

In the first subsection of Section 3, we propose a function field analogue of Carmichael's totient function conjecture, and prove that in contrast to the classical case, one can answer the function field analogue of Carmichael's conjecture in the negative. In the last subsection of Section 3, we propose a function field analogue of Sierpiński's conjecture, and discuss some special cases of this analogue.

2. A Carlitz module analogue of Euler's totient function

In this section, we introduce a Carlitz module analogue of Euler's totient function, and prove an analogue of the Euler theorem for the Carlitz module.

We recall the Carlitz module analogue of the Fermat little theorem.

Lemma 2.1. *(See Hayes [9, Proposition 2.4].) Let $\wp$ be a monic prime in $\mathbf{A}$, and let u be an element in $\mathbf{A}$. Then*

$$C_{\wp-1}(u) \equiv 0 \pmod{\wp}.$$