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On number of partitions of an integer into a fixed number of positive integers

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ABSTRACT

Text. This paper focuses on the number of partitions of a positive integer n into k positive summands, where k is an integer between 1 and n. Recently some upper bounds were reported for this number in [Merca14]. Here, it is shown that these bounds are not as tight as an earlier upper bound proved in [Andrews76-1] for $k \leq 0.42n$. A new upper bound for the number of partitions of n into k summands is given, and shown to be tighter than the upper bound in [Merca14] when k is between $O(\frac{\sqrt{n}}{\ln n})$ and $n - O(\frac{\sqrt{n}}{\ln n})$. It is further shown that the new upper bound is also tighter than two other upper bounds previously reported in [Andrews76-1] and [Colman82]. A generalization of this upper bound to number of partitions of n into at most k summands is also presented.

Video. For a video summary of this paper, please visit http://youtu.be/Pb6lKB3MnME.

1. Introduction

Partitions of an integer play an important role in the solutions of combinatorial problems and this article is motivated in part by such a problem that arises in counting

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multicast calls between n callers and n receivers in a switching network [Oruc15]. In particular, three types of partitions will be of interest in this paper as stated below and we refer the reader to [Andrews76-1] for basic concepts in partition theory.

1. A partition of n is an unordered sum of n that comprises up to n positive integers. The number of such sums is often denoted by $p(n)$. For example, $p(3) = 3$ as $3 = 3$, $3 = 1 + 2$, and $3 = 1 + 1 + 1$.
2. A partition of n into exactly k parts is an unordered sum of n that uses exactly k positive integers. The number of such partitions will henceforth be denoted by $p(n, k)$. For example, $p(5, 3) = 2$ as $5 = 1 + 2 + 2$ and $5 = 1 + 1 + 3$ are the only two sums of 5 that can be formed using three positive integers.
3. A partition of n into at most k parts is an unordered sum of n that uses at most k positive integers. Following the asterisk notation in [Colman82], the number of such partitions will henceforth be denoted by $p^*(n, k)$. For example, $p^*(5, 3) = 5$ as $5 = 5$, $5 = 2 + 3$, $5 = 4 + 1$, $5 = 2 + 2 + 1$, and $5 = 3 + 1 + 1$, are the only sums of 5 that can be formed using one, two, or three positive integers.

No exact closed-form expressions are known to compute the values of $p(n)$, $p(n, k)$, and $p^*(n, k)$. For $p(n)$, Hardy–Ramanujan–Rademacher formula provides an asymptotic approximation to $p(n)$ [Andrews76-2]:

$$p(n) \approx \frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{\frac{2n}{3}}}. \tag{1}$$

Using Remark 1 in [Kane06], it can be shown that

$$0.02556 \leq \lim_{n\to\infty} \frac{p(n)}{\frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{\frac{2n}{3}}}} \leq 37.6393, \tag{2}$$

while

$$\lim_{n\to\infty} \left| \frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{\frac{2n}{3}}} - p(n) \right| = \infty. \tag{3}$$

In the sequel, we will also need Kane's inequality:

$$\frac{C^-_{1,1}}{n} e^{\pi\sqrt{\frac{2n}{3}}} \leq p(n) \leq \frac{C^+_{1,1}}{n} e^{\pi\sqrt{\frac{2n}{3}}}, \tag{4}$$

where $C^-_{1,1}$ is any number less than $\frac{5e^{-2-\frac{3\gamma}{2}}}{8\sqrt{3}\pi^{3/2}}$, $C^+_{1,1}$ is any number greater than $\frac{27}{4}\left(\frac{e}{\pi}\right)^{3/2}$, and $\gamma = 0.57721\ldots$, is the Euler constant. We will let $C^-_{1,1} = 0.0036$ and $C^+_{1,1} = 5.44$.

In the remainder of the paper, we analyze the previously reported upper bounds for $p(n, k)$ and present a sharper upper bound for the same. This new bound is used to