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# Rational period functions in higher level cases

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ABSTRACT

Extending Knopp's results [9,10] we investigate examples and properties of rational period functions in higher level cases, including location of poles and behavior under the action of Hecke operators. More precisely, we prove that a rational period function may have poles only at 0 or at real quadratic irrationalities. Moreover by applying the action of Hecke operators we prove that for positive odd integer $k$ and $p \in \{2, 3\}$, the space of all rational period functions of weight $2k$ for $\Gamma_0^+(p)$ is infinite dimensional.


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## 1. Introduction and statement of results

For $k \in \mathbb{Z}$ and any meromorphic function $f$ on the complex upper half plane $\mathfrak{H}$, we define the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ by

$$(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z).$$

Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $W_p = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$, and $U = TW_p = \left(\begin{smallmatrix} \sqrt{p} & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$. Duke, Imamoḡlu, and Tóth [6] (for $p = 1$) and we [3,4] (for $p \in \{2,3\}$) have studied a polynomial $q(z)$ called *period polynomial* satisfying

$$q|_{2k}W_p + q = 0 \tag{1}$$

and

$$q|_{2k}U^{n_p-1} + q|_{2k}U^{n_p-2} + \cdots + q|_{2k}U + q = 0, \tag{2}$$

where $k < 0$ and $n_p = \begin{cases} 3, & \text{if } p = 1, \\ 2p, & \text{if } p = 2, 3. \end{cases}$ We denote by $W_{-2k}(\Gamma_0^+(p))$ the set of all such polynomials. On the other hand, a rational function $q(z)$ satisfying (1) and (2) is called a *rational period function of weight* $2k$ *for* $\Gamma_0^+(p)$. We denote by $\mathrm{RPF}_{2k}(\Gamma_0^+(p))$ the set of all such functions. It is natural to investigate the difference between period polynomials and rational period functions. The notion of rational period functions was initiated by Knopp [9] who showed that when $p = 1$ and $k$ is odd, the function

$$q_{2k}(z) := (z^2 - z - 1)^{-k} + (z^2 + z - 1)^{-k}$$

belongs to the space $\mathrm{RPF}_{2k}(\Gamma_0^+(1))$. As we see in $q_{2k}(z)$, a rational period function may have poles unlike period polynomials. Duke, Imamoḡlu, and Tóth [6] (for $p = 1$) and we [3,4] (for $p \in \{2,3\}$) have found that the space $W_{-2k}(\Gamma_0^+(p))$ is finite dimensional. In this paper we deal with a problem whether or not the space $\mathrm{RPF}_{2k}(\Gamma_0^+(p))$ is finite dimensional. As for the location of poles of an arbitrary rational period function $q(z)$, Knopp [10] (for $p = 1$) proved that $q(z)$ can have poles only at 0 or at real quadratic irrationalities. In the next two theorems we extend Knopp's two results mentioned above to the higher level cases.

**Theorem 1.1.** *Let* $p \in \{2,3\}$ *and* $k$ *be an odd integer. We put*

$$z_0 = \frac{p + \sqrt{p^2 + 4p}}{2p}, \quad z_0' = \frac{1}{pz_0} = \frac{-p + \sqrt{p^2 + 4p}}{2p}$$

*and define*