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Sets characterized by missing sums and differences in dilating polytopes ☆

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ABSTRACT

Text. A sum-dominant set is a finite set A of integers such that $|A + A| > |A - A|$. As a typical pair of elements contributes one sum and two differences, we expect sum-dominant sets to be rare in some sense. In 2006, however, Martin and O'Bryant showed that the proportion of sum-dominant subsets of $\{0, \ldots, n\}$ is bounded below by a positive constant as $n \to \infty$. Hegarty then extended their work and showed that for any prescribed $s, d \in \mathbb{N}_0$, the proportion $\rho_n^{s,d}$ of subsets of $\{0, \ldots, n\}$ that are missing exactly s sums in $\{0, \ldots, 2n\}$ and exactly $2d$ differences in $\{-n, \ldots, n\}$ also remains positive in the limit.

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More sum than difference sets
Convex sets

We consider the following question: are such sets, characterized by their sums and differences, similarly ubiquitous in higher dimensional spaces? We generalize the integers in a growing interval to the lattice points in a dilating polytope. Specifically, let P be a polytope in $\mathbb{R}^D$ with vertices in $\mathbb{Z}^D$, and let $\rho_n^{s,d}$ now denote the proportion of subsets of $L(nP)$ that are missing exactly s sums in $L(nP)+L(nP)$ and exactly $2d$ differences in $L(nP)-L(nP)$. As it turns out, the geometry of P has a significant effect on the limiting behavior of $\rho_n^{s,d}$. We define a geometric characteristic of polytopes called local point symmetry, and show that $\rho_n^{s,d}$ is bounded below by a positive constant as $n \to \infty$ if and only if P is locally point symmetric. We further show that the proportion of subsets in $L(nP)$ that are missing exactly s sums and at least $2d$ differences remains positive in the limit, independent of the geometry of P. A direct corollary of these results is that if P is additionally point symmetric, the proportion of sum-dominant subsets of $L(nP)$ also remains positive in the limit.

Video. For a video summary of this paper, please visit http://youtu.be/2M8Qg0E0RAc.

Contents

1. Introduction

Given a finite set $A \subset \mathbb{Z}$, we define the sumset $A+A$ and the difference set $A-A$ by

$$\begin{aligned} A+A &= \{a_1+a_2 : a_1, a_2 \in A\}, \\ A-A &= \{a_1-a_2 : a_1, a_2 \in A\}. \end{aligned} \tag{1.1}$$

It is natural to compare the sizes of $A+A$ and $A-A$ as we vary A over a family of sets. As addition is commutative while subtraction is not, a pair of distinct elements $a_1, a_2 \in A$ generates two differences a_1-a_2 and a_2-a_1 but only one sum a_1+a_2. We thus expect that most of the time, the size of the difference set is greater than that of the sumset—that is, we expect most sets A to be *difference-dominant*. It is possible, however, to construct sets whose sumsets have more elements than their difference sets. Such sets are called *sum-dominant* or *More Sums Than Differences* (MSTD) sets. The first example of an