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New congruences modulo 5 for the number of 2-color partitions

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ABSTRACT

Let $p_k(n)$ be the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of k. In this paper, we find some interesting congruences modulo 5 for $p_k(n)$ for $k \in \{2, 3, 4\}$ by employing Ramanujan's theta function identities and some identities for the Rogers–Ramanujan continued fraction. The congruence for $p_2(n)$ was earlier proved by Chen and Lin with the aid of modular forms.

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1. Introduction

A partition of a natural number n is a finite sequence of non-increasing positive integer parts where the sum of the parts is equal to n. If $p(n)$ denote the number of partitions of n, then

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, as customary, for any complex number a and $|q| < 1$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

Ramanujan's so-called "most beautiful identity" for the partition function $p(n)$ is

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

which readily implies one of his three famous partition congruences, namely,

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.2}$$

In particular, we have

$$p(25n+24) \equiv 0 \pmod 5. \tag{1.3}$$

A stronger version

$$p(25n+24) \equiv 0 \pmod{25}$$

can also be deduced easily from (1.1) (see [4, p. 38]).

Now, let $p_0(n) := p(n)$ and for a positive integer k, define integers $p_k(n)$ by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty (q^k;q^k)_\infty}.$$

Note that $p_k(n)$ is the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of k.

For $k = 1$, it is known that [2, eq. (5.4)]

$$p_1(25n+23) \equiv 0 \pmod 5, \tag{1.4}$$

which can also be shown to be true for modulo 25.