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Explicit examples for the Breuil–Mézard conjecture

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ABSTRACT

In this paper, we compute some universal deformation rings for certain rank two Galois representations. We then study the relations between different deformation rings. These relations give explicit examples for the Breuil–Mézard conjecture.

1. Introduction

Fix a prime number $p \geq 3$, a finite extension $\mathbb{F}/\mathbb{F}_p$. Let $\bar{r} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a rank two continuous representation. In this paper, for certain non-split $\bar{r}$, we compute a Fontaine–Laffaille deformation ring of $\bar{r}$ and a potentially Barsotti–Tate deformation ring of $\bar{r}$. We then construct a canonical isomorphism between these two deformation rings, which provides an explicit example for the geometric Breuil–Mézard conjecture.

1.1. *The geometric Breuil–Mézard conjecture*

Let L be a finite totally ramified extension of $W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}$ and uniformiser π. We assume that L is sufficiently large, in particular, we assume that

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$\sharp\mathbb{F} > 5$, so that $\mathrm{PGL}_2(\mathbb{F})$ is a simple group. (Note that this condition is not essential for the computation in this paper, but it is needed in [7] to prove Theorem 1.3 and in Section 4.1 for the global situation.) Let $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(L)$ be an inertial type, i.e., a representation with open kernel which extends to $W_{\mathbb{Q}_p}$. Let ϵ and ω be the p-adic cyclotomic character and the mod p cyclotomic character respectively. Fix integers a, b with $b \geq 0$ and a character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\overline{\psi\epsilon} = \det \bar{r}$. We let $R^{\square,\psi}(a,b,\tau,\bar{r})$ and $R^{\square,\psi}_{cr}(a,b,\tau,\bar{r})$ be the framed deformation $\mathcal{O}$-algebras which are universal for framed deformations of $\bar{r}$ with determinant $\psi\epsilon$, and are potentially semistable (respectively potentially crystalline) with Hodge–Tate weights $(a, a+b+1)$ and inertia type τ. Let $\sigma(\tau)$ and $\sigma^{cr}(\tau)$ denote the finite-dimensional irreducible L-representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ corresponding to τ via Henniart's inertial local Langlands correspondence. We set $\sigma(a,b,\tau) = \sigma(\tau) \otimes_L \det^a \mathrm{Symm}^b L^2$ and $\sigma^{cr}(a,b,\tau) = \sigma^{cr}(\tau) \otimes_E \det^a \mathrm{Symm}^b L^2$. We let $L_{a,b,\tau}$ (respectively $L^{cr}_{a,b,\tau}$) be a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice in $\sigma(a,b,\tau)$ (respectively $\sigma^{cr}(a,b,\tau)$). Write $\sigma_{m,n}$ for the representation $\det^m \otimes \mathrm{Symm}^n \mathbb{F}^2$ of $\mathrm{GL}_2(\mathbb{F}_p)$, $0 \leq m \leq p-2$, $0 \leq n \leq p-1$. Then we may write

$$(L_{a,b,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \oplus_{m,n} \sigma_{m,n}^{a_{m,n}},$$

and

$$(L^{cr}_{a,b,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \oplus_{m,n} \sigma_{m,n}^{a^{cr}_{m,n}},$$

for some integers $a_{m,n}$ and $a^{cr}_{m,n}$. The geometric Breuil–Mézard conjecture is the following.

Conjecture 1.1. *There are cycles $\mathcal{C}_{m,n}(\bar{r})$ depending only on m, n, and $\bar{r}$ such that for any a, b, τ,*

$$Z(R^{\square,\psi}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a_{m,n}\mathcal{C}_{m,n}(\bar{r})$$

and

$$Z(R^{\square,\psi}_{cr}(a,b,\tau,\bar{r})/\pi) = \sum_{m,n} a^{cr}_{m,n}\mathcal{C}_{m,n}(\bar{r}).$$

Remark 1.2. See [7, Section 1.1] for the definition of cycles. As remarked in [7, Remark 3.1.5], or by [7, Lemma 4.3.1], the truth of the conjecture is independent of the choice of ψ. We may assume that ψ is crystalline. If Conjecture 1.1 is true for all a, b, τ, then $\mathcal{C}_{m,n}(\bar{r}) = 0$ unless $\det \bar{r}|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$. Furthermore, if $\det \bar{r}|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$, we must have

$$\mathcal{C}_{m,n}(\bar{r}) = Z(R^{\square,\psi}_{cr}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi),$$

where $\tilde{m}$ is chosen so that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{2\tilde{m}+n}$ and $\tilde{m} \equiv m \pmod{p-1}$.