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Dirichlet's eta and beta functions: Concavity and fast computation of their derivatives

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A B S T R A C T

For any $a \in (0, \infty)$, we prove the strict concavity of the function

$$\eta_a(t) := \sum_{m=0}^{\infty} \frac{(-1)^m}{(am+1)^t}$$

on $(0, \infty)$, and provide fast computations of their derivatives on $(0, \infty)$. We give short proofs mainly based on differentiation formulas concerning the gamma process. In particular, our results apply to Dirichlet's eta and beta functions $\eta_1(t)$ and $\eta_2(t)$, respectively.

1. Introduction and main results

For any $a \in (0, \infty)$, we consider the function

$$\eta_a(t) := \sum_{m=0}^{\infty} \frac{(-1)^m}{(am+1)^t}, \qquad t \in (0, \infty). \tag{1}$$

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For complex numbers t with positive real part, the functions $\eta_1(t)$ and $\eta_2(t)$ are known in the literature as Dirichlet's eta and beta functions, respectively. Both functions play an important role in analytic number theory and mathematical physics, among other fields. Integral and series representations for η_1 and η_2 can be found in many papers. See, for instance, those in Sondow [14] referring to η_1, in Guillera and Sondow [10] concerning the Lerch transcendent function, which includes η_1 and η_2 as particular cases, or in Cvijović [9] referring to the Legendre chi function, containing η_2 as a special case. On the other hand, fast computations of η_1 by means of Chebyshev and other polynomials $p_n(x)$ satisfying the condition $p_n(-1) \neq 0$ can be found in Borwein [5], whereas fast convergent series for $\eta_2(n)$, $n = 1, 2, \dots$, using Markov–Wilf–Zeilberger theory are given in [11].

The starting point of this note is a recent paper by Alzer and Kwong [4] showing that Dirichlet's eta function η_1 is strictly concave on $(0, \infty)$, that is,

$$\lambda\eta_1(x) + (1-\lambda)\eta_1(y) < \eta_1(\lambda x + (1-\lambda)y),$$

for any $x, y > 0$ with $x \neq y$, and $\lambda \in (0, 1)$. In this way, Alzer and Kwong [4] improve a previous result by Wang [15], in which this author shows that η_1 is strictly logarithmically concave on $(0, \infty)$. Here, we provide a short proof of the strict concavity of η_a on $(0, \infty)$, for any $a \in (0, \infty)$. This proof is mainly based on the differential calculus for the gamma process developed in [1]. Such a calculus is also used to give fast computations of the derivatives $\eta_a^{(k)}$, $k = 0, 1, \dots$, on $(0, \infty)$.

The main results of this note are the following.

Theorem 1.1. *For any $a \in (0, \infty)$, the function η_a is strictly concave on $(0, \infty)$. Moreover,*

$$\lim_{t\to 0} \eta_a(t) = \frac{1}{2}, \qquad \lim_{t\to\infty} \eta_a(t) = 1.$$

Let $\mathbb{Z}_+$ be the set of nonnegative integers and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$. The mth forward differences of any function $f : \mathbb{Z}_+ \to \mathbb{R}$ are recursively defined by $\Delta^0 f(l) = f(l)$, $\Delta^1 f(l) = f(l+1) - f(l)$, $l \in \mathbb{Z}_+$, and for any $m = 2, 3, \dots$

$$\Delta^m f(l) := \Delta^1(\Delta^{m-1} f)(l) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(l+j), \qquad l \in \mathbb{Z}_+. \tag{2}$$

The second equality in (2) can be shown by induction on m (cf. Jordan [13, p. 8]). For any $a, t \in (0, \infty)$ and $k \in \mathbb{Z}_+$, denote by

$$\phi_{a,t,k}(l) := \frac{\log^k(al+1)}{(al+1)^t}, \qquad l \in \mathbb{Z}_+. \tag{3}$$

The following result gives fast computations of the derivatives of η_a.