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Bounds on the number of Diophantine quintuples

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ABSTRACT

We consider Diophantine quintuples $\{a, b, c, d, e\}$. These are sets of distinct positive integers, the product of any two elements of which is one less than a perfect square. It is conjectured that there are no Diophantine quintuples; we improve on current estimates to show that there are at most $2.3 \cdot 10^{29}$ Diophantine quintuples.

1. Introduction

Consider the set $\{1, 3, 8, 120\}$. This has the property that the product of any two of its elements is one less than a square. Define a Diophantine m-tuple as a set of m integers $\{a_1, \ldots, a_m\}$ with $a_1 < a_2 < \ldots < a_m$, such that $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$. Throughout the rest of this article we simply refer to m-tuples, and not to Diophantine m-tuples.

One may extend any triple $\{a, b, c\}$ to a quadruple $\{a, b, c, d_+\}$ where

$$d_+ = a + b + c + 2abc + 2rst, \quad r = \sqrt{ab+1}, \quad s = \sqrt{ac+1}, \quad t = \sqrt{bc+1}, \qquad (1)$$

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Table 1
Bounds on the number of quintuples.

	Upper bound on number of quintuples
Dujella [10]	10^{1930}
Fujita [16]	10^{276}
Filipin and Fujita [12]	10^{96}
Elsholtz, Filipin and Fujita [11]	$6.8 \cdot 10^{32}$
Cipu [6]	10^{31}
Trudgian	$2.3 \cdot 10^{29}$

by appealing to a result by Arkin, Hoggatt and Straus [2]. Indeed, they conjectured that *every* such quadruple is formed in this way. We record this in

Conjecture 1 *(Arkin, Hoggatt and Straus). If $\{a, b, c, d\}$ is a quadruple then $d = d_+$.*

Note that any possible quintuple $\{a, b, c, d, e\}$ contains, *inter alia* the quadruples $\{a, b, c, d\}$ and $\{a, b, c, e\}$. If Conjecture 1 is true then $d_+ = d = e$, whence d and e are not distinct. Therefore Conjecture 1 implies

Conjecture 2. *There are no quintuples.*

Dujella [10] proved that there are finitely many quintuples. Subsequent research, summarised in Table 1, has reduced the bound on the total number of quintuples. We prove

Theorem 1. *There are at most $2.3 \cdot 10^{29}$ quintuples.*

Wu and He [27] did not estimate the number of quintuples, though bounds for the second largest element d were considered in some special cases – see Section 3 for more details. We also note that the proof of Proposition 4.2 in [16] appears to be flawed, and hence the estimate in [11] is too small. We repair the proof, and improve on it slightly, in Section 4.

The layout of the paper is as follows. In Section 2 we define several classes of quintuples and identify doubles and triples that cannot be extended to quintuples. In Sections 3 and 4 we bound the size of the second largest element of a quintuple. Essential to Dujella's argument, and to all subsequent improvements, is a result by Matveev [23] on linear forms of logarithms. We make use of a result by Aleksentsev [1] which, for our purposes, is slightly better. In several places we optimise the argument given by Fujita [16].

In Section 5 we estimate some sums from elementary number theory. In Section 6 we estimate the total number of quintuples, and we prove Theorem 1. In Section 7 we define $D(-1)$-quadruples and, using one of our ancillary results, make a small improvement on the estimated number of these. In Section 8 we conclude with some ideas on possible future improvements.