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Quasi-modular forms attached to elliptic curves: Hecke operators

Hossein Movasati

Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil

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A B S T R A C T

In this article we describe Hecke operators on the differential algebra of geometric quasi-modular forms. As an application for each natural number d we construct a vector field in six dimensions which determines uniquely the polynomial relations between the Eisenstein series of weight 2, 4 and 6 and their transformation under multiplication of the argument by d, and in particular, it determines uniquely the modular curve of degree d isogenies between elliptic curves.

1. Introduction

The theory of quasi-modular forms was first introduced by Kaneko and Zagier in [5] due to its applications in mathematical physics. One can describe quasi-modular forms in the framework of the algebraic geometry of elliptic curves, and in particular, the Ramanujan differential equation between Eisenstein series can be derived from the Gauss–Manin connection of families of elliptic curves, see for instance [7] and [9]. We call this the Gauss–Manin connection in disguise. The terminology arose from a private letter of

E-mail address: hossein@impa.br.
URL: http://www.impa.br/~hossein.

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Pierre Deligne to the author, see [4]. In the present article we describe Hecke operators for such quasi-modular forms, and certain differential ideals related to modular curves. In [3] the authors describe a differential equation in the j-invariant of two elliptic curves which is tangent to all modular curves of degree d isogenies of elliptic curves. This differential equation can be derived from the Schwarzian differential equation of the j-function and the latter can be calculated from the Ramanujan differential equation between Eisenstein series. This suggests that there must be a relation between Ramanujan differential equation and modular curves. In this article we also establish this relation. Another motivation behind this work is to prepare the ground for similar topics in the case of Calabi–Yau varieties, see [8].

Consider the Ramanujan ordinary differential equation

$$\mathsf{R}: \begin{cases} \dot{s}_1 = \frac{1}{12}(s_1^2 - s_2) \\ \dot{s}_2 = \frac{1}{3}(s_1 s_2 - s_3) \\ \dot{s}_3 = \frac{1}{2}(s_1 s_3 - s_2^2) \end{cases} \dot{s}_k = \frac{\partial s_k}{\partial \tau} \tag{1}$$

which is satisfied by the Eisenstein series:

$$s_i(\tau) = a_i E_{2i}(q) := a_i \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \\ i = 1, 2, 3, \ q = e^{2\pi i \tau}, \ \mathrm{Im}(\tau) > 0 \tag{2}$$

and

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (2\pi i, (2\pi i)^2, (2\pi i)^3).$$

The algebra of modular forms for $\mathrm{SL}(2, \mathbb{Z})$ is generated by the Eisenstein series E_4 and E_6 and all modular forms for congruence groups are algebraic over the field $\mathbb{C}(E_4, E_6)$, see for instance [14]. In a similar way the algebra of quasi-modular forms for $\mathrm{SL}(2, \mathbb{Z})$ is generated by E_2, E_4 and E_6, see for instance [6,9], and we have:

Theorem 1. *For $i = 1, 2, 3$ and $d \in \mathbb{N}$, there is a homogeneous polynomial $I_{d,i}$ of degree $i \cdot \psi(d)$, where $\psi(d) := d \prod_p (1 + \frac{1}{p})$ is the Dedekind ψ function and p runs through primes p dividing d, in the weighted ring*

$$\mathbb{Q}[t_i, s_1, s_2, s_3], \ \mathrm{weight}(t_i) = i, \ \mathrm{weight}(s_j) = j, \ j = 1, 2, 3 \tag{3}$$

and monic in the variable t_i such that $t_i(\tau) := d^{2i} \cdot s_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)$ satisfy the algebraic relation:

$$I_{d,i}(t_i, s_1, s_2, s_3) = 0.$$

Moreover, for $i = 2, 3$ the polynomial $I_{d,i}$ does not depend on s_1.