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Distribution of integral lattice points in an ellipsoid with a diophantine center

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We evaluate the mean square limit of exponential sums related to a rational ellipsoid, extending a work of Marklof. Moreover, as a result of it, we study the asymptotic values of the normalized deviations of the number of lattice points inside a rational ellipsoid and inside a rational thin ellipsoidal shell.

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1. Introduction

Let $\mathfrak{S}_n^+(\mathbb{Q})$ be the family of all positive definite and symmetric $n \times n$ matrices with rational components. Given $\mathrm{M} \in \mathfrak{S}_n^+(\mathbb{Q})$, we consider the quadratic form Q_{M} defined by $Q_{\mathrm{M}}(\mathbf{x}) = \langle \mathrm{M}\mathbf{x}, \mathbf{x}\rangle$ for $\mathbf{x} \in \mathbb{R}^n$ and the corresponding ellipsoid

$$E_R^{\mathrm{M}}(\boldsymbol{\alpha}) = \{\mathbf{x} \in \mathbb{R}^n : Q_{\mathrm{M}}(\mathbf{x} - \boldsymbol{\alpha}) \leq R^2\}$$

centered at a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, and we write $E_R^{\mathrm{M}} = E_R^{\mathrm{M}}(\mathbf{0})$ for $R > 0$.

Our interests are focused on the distribution of integral lattice points inside E_R^{M} as R tends to infinity. In place of the ellipsoid centered at the origin, we consider the ellipsoid with a diophantine center of type κ as defined below.

Definition 1.1. A vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ is said to be *of diophantine type* κ, if there exists a constant $c_0 > 0$ such that $\left|\boldsymbol{\alpha} - \dfrac{\mathbf{m}}{q}\right| > \dfrac{c_0}{q^\kappa}$ for all $\mathbf{m} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$.

The smallest possible value of κ is $1 + 1/k$ in the above definition. In this case, $\boldsymbol{\alpha}$ is called *badly approximable* (see [8]).

We consider the counting function N_{M} in the ellipsoid with a diophantine center of type κ introduced in [1]. For convenience, we assume that $\mathrm{M} \in \mathfrak{S}_n^+(\mathbb{Z})$ for now and we shall see that this can be extended to the case $\mathrm{M} \in \mathfrak{S}_n^+(\mathbb{Q})$ as noted in Remark 1.6. Let $\mathbb{1}_B$ be the characteristic function of the unit open ball $B = B_1$ in $\mathbb{R}^n$ and $\mathbb{1}_{E^{\mathrm{M}}}$ be the characteristic function of the ellipsoid $E^{\mathrm{M}} = E_1^{\mathrm{M}}$ corresponding to $\mathrm{M} \in \mathfrak{S}_n^+(\mathbb{Z})$. For $t > 0$, we denote by $N_{\mathrm{M}}(t) := \sharp\big(\mathbb{Z}^n \cap E_t^{\mathrm{M}}(\boldsymbol{\alpha})\big)$ the number of lattice points inside the ellipsoid $E_t^{\mathrm{M}}(\boldsymbol{\alpha})$ centered at a diophantine vector $\boldsymbol{\alpha} \in \mathbb{R}^n$; that is to say,

$$N_{\mathrm{M}}(t) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \mathbb{1}_{E^{\mathrm{M}}}\left(\frac{\mathbf{m} - \boldsymbol{\alpha}}{t}\right).$$

In this paper, we investigate the asymptotics of the following deviations: from their asymptotics (see (2.3) and (2.4)), we can consider the normalized deviation $F_{\mathrm{M}}(t)$ of $N_{\mathrm{M}}(t)$ defined by

$$F_{\mathrm{M}}(t) := \frac{N_{\mathrm{M}}(t) - |E^{\mathrm{M}}|\, t^n}{t^{(n-1)/2}} \quad \text{as } t \to \infty \tag{1.1}$$

and the normalized deviation $S_{\mathrm{M}}(t, \eta)$ of the number of lattice points inside the spherical shell between the elliptic spheres of radii $t + \eta$ and t given by

$$S_{\mathrm{M}}(t, \eta) := \frac{N_{\mathrm{M}}(t + \eta) - N_{\mathrm{M}}(t) - |E^{\mathrm{M}}|((t + \eta)^n - t^n)}{\sqrt{\eta}\, t^{(n-1)/2}} \tag{1.2}$$

as $t \to \infty$ and $\eta \to 0$, where $|E^{\mathrm{M}}|$ denotes the volume of the ellipsoid E^{M}.