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The distribution of central values of elliptic curve L-functions ☆

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We present some numerical results on the distribution of nonzero central values of elliptic curve L-functions. We compare the family of elliptic curves $E_{a,b} : y^2 = x^3 + ax + b$ with (a, b) ranging over a large box, and Cremona's list of all elliptic curves with conductor $340\,000 \leq N < 350\,000$. We also present a non-rigorous method for calculating central values that is faster than the approximate functional equation by a logarithmic factor.

1. Overview

The distribution of ranks of rational elliptic curves has been a well-studied problem. Since the literature on this topic is vast, we refer to the nice survey article [BMSW] for

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a description of this problem and for its references. Since the publication of [BMSW], it is now known that the average rank of elliptic curves is bounded by work of Bhargava and Shankar [BS]. Our motivation in this paper is to study numerically the central value distribution of rank 0 elliptic curve L-functions. We have found that the distribution is apparently rather sensitive to the ordering of the family. In order to calculate central values of elliptic curve L-functions quickly (and thereby study larger families and/or curves with larger conductors), we investigated ways to speed up the calculations. We shall present a non-rigorous method that, in our data set, is faster than the approximate functional equation by a logarithmic factor.

2. The families

There are a few natural ways to order the family of all rational elliptic curves. Probably the most natural is by conductor. However, this ordering is not very explicit, and it is not even known how to asymptotically count the number of elliptic curves with conductor $N \leq X$. One can also order by the absolute value of the discriminant, $|\Delta|$, but this has the same previously-mentioned problems as ordering by conductor. For theoretical purposes, the most well-studied ordering has been to take Weierstrass equations $E_{a,b}$: $y^2 = x^3 + ax + b$ with a and b in certain growing intervals. This family eventually captures every rational elliptic curve, but it may take a long time to find cases where there is cancellation between $4a^3$ and $27b^2$, or cases where the conductor is much smaller than the discriminant.

In a given family $\mathcal{F}$, we shall study the collection of central values $\{L(1/2, E) : E \in \mathcal{F}\}$ (we normalize the L-function so 1/2 is the center of the critical strip). We shall also restrict attention to families where the root number is +1, since all central values vanish otherwise.

We focus on two particular families. One, which we denote $\mathcal{F}_{\text{Box}}$, is the set of curves $E_{a,b}$ with $150 \leq a \leq 900$ and $700 \leq b \leq 10\,000$. Here $|\Delta| \leq 9.0 \times 10^{10}$ with conductor generally not significantly smaller than $|\Delta|$ (so usually of order 10^{10}). In addition, we discard a and b if the Weierstrass equation is not minimal; we are left with 6 957 450 curves. For each curve E, we used PARI [P] to compute the root number w_E, a_n's (here $a_p = p + 1 - \#E(\mathbb{F}_p)$), conductor N, Tamagawa numbers c_p, and real period Ω (i.e., all the standard BSD data except for the central value and $|\text{Ш}|$). We calculated a numerical approximation for the central value $L(1/2, E)$ using a custom method that is many times faster than PARI's default central value algorithm; we present our method in Section 4. However, the method is not rigorous because we are unable to bound the error in the approximation, yet we do have strong evidence that the method works. The data from $\mathcal{F}_{\text{Box}}$ were computed in Summer 2005 on a Dell Inspiron 6000 with a 1.6 GHz Pentium M and 1 GB of RAM, and took approximately 1100 hours.

Among the 6 957 450 curves, 3 476 381 have odd rank and 3 481 069 have even rank (for the sake of curiosity notice that their difference is 4688, while $\sqrt{6\,957\,450} \approx 2638$). Among the even rank curves, 1 262 463 are probabilistically (meaning, using our approximation