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Computation of framed deformation functors

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ABSTRACT

In this work we compute the framed deformation functor associated to a reducible representation given as a direct sum of 2-dimensional representations associated to elliptic curves with appropriate local conditions. Such conditions arise in the works of Schoof and correspond to reduction properties of modular elliptic curves.

1. Introduction

This work originates from the articles of Schoof about classification of abelian varieties [11]. There he examines the case of abelian varieties over $\mathbb{Q}$ with semistable reduction in only one prime ℓ and good reduction everywhere else, and proves that they do not exist for $\ell = 2, 3, 5, 7, 13$, while for $\ell = 11$ they are isogenous to a product of the Jacobian variety $J_0(11)$ of the modular curve $X_0(11)$. In [12] he makes some generalizations of this result when ℓ is not a prime and the base field is not $\mathbb{Q}$, but a quadratic field. Some similar results, given in terms of p-divisible groups, were also previously obtained by Abrashkin in [1].

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The main purpose of this work is to translate some results of those articles in terms of deformation theory of representations associated to elliptic curves. We examine the following setting: let $p \neq \ell$ be distinct primes and $S = \{p, \ell, \infty\}$. Let $\bar{\rho}_i : G_S \to GL_2(k)$, $i = 1, \dots, n$, be Galois representations, where k is a finite field of characteristic p and G_S is the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. We can suppose that there are exactly r non-isomorphic representations among them and, up to reordering indices, suppose that they are $\bar{\rho}_1, \dots, \bar{\rho}_r$. Then we can write

$$\bar{\rho} = \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_n = \oplus_{i=1}^r \bar{\rho}_i^{e_i} : G_S \to GL_{2n}(k) \tag{1}$$

where $\sum_{i=1}^r e_i = n$. The main result is the following:

Theorem 1.1. *Suppose that:*

1. *the k-vector space $Ext^1_{\underline{D},p}(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$ of killed-by-p extensions is trivial for every $i, j = 1, \dots, r$;*
2. *$Hom_{G_S}(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$*

Let $F^{\square}_{\underline{D}}$ be the framed deformation functor associated to $\bar{\rho}$ with the local conditions:

- *ρ is p-flat over $\mathbb{Z}[1/\ell]$;*
- *ρ satisfies $(\rho_i(g) - Id)^2 = 0$ for every $g \in I_\ell$;*
- *ρ is odd.*

Then $F^{\square}_{\underline{D}}$ is represented by a framed universal ring $R^{\square}_{\underline{D}}$ which is isomorphic to $W(k)[[x_1, \dots, x_N]]$, where $N = 4n^2 - \sum_{i=1}^r {e_i}^2$.

The setting works in particular when $\overline{\rho_i}$ is the representation associated to the p-torsion points of an elliptic curve E_i over $\mathbb{Q}$ having semistable reduction in ℓ and good supersingular reduction at p, as the varieties described in [11]. Moreover, the local condition in ℓ corresponds to the condition of semistable action described in [11, Section 2], while the condition in p is the classical flatness condition introduced in [10]. The final result wants to express that the framed deformation ring turns out to be the "simplest" possible.

The Main Theorem was mainly thought to give an analog for deformation of [11, Th. 8.3]. In order to give a "deformational" proof of Schoof's result, we should show that the universal deformation of a representation associated to a p-divisible group that can be written as a direct sum of copies of a simple group is given by a power of the Tate Module. Anyway this cannot be proved, since the representation is not absolutely irreducible and we have no guarantee that such a universal representation exists. Therefore we decided to shift to analyze a framed deformation problem, to avoid representability issues, and to work in a more general setting: we do not require our representation to