Journal of Number Theory 156 (2015) 52–74

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the saturation number for cubic surfaces

Yuchao Wang

School of Mathematics, Shandong University, Jinan, 250100, China

ARTICLE INFO

Article history:
Received 3 August 2014
Accepted 30 March 2015
Available online 12 May 2015
Communicated by Robert C. Vaughan

MSC:
11D25
11P32
11P55
11N36

Keywords:
Sieve method
Circle method
Universal torsors

ABSTRACT

We investigate the density of rational points on the Fermat cubic surface and the Cayley cubic surface whose coordinates have few prime factors. The key tools used are the weighted sieve, the circle method and universal torsors.

1. Introduction

In this paper we are concerned with the almost prime integral points on cubic surfaces.

Let P_r indicate an r-almost prime, which is a number with at most r prime factors, counted with multiplicity. Furthermore, let $\mathbb{Z}^4_{\text{prim}}$ be the set of vectors $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ with $\gcd(x_0, x_1, x_2, x_3) = 1$. For any cubic surface $S \subset \mathbb{P}^3$ defined

E-mail address: yuchaowang@mail.sdu.edu.cn.

http://dx.doi.org/10.1016/j.jnt.2015.03.014

over $\mathbb{Q}$, we define the saturation number $r(S)$ to be the least number r such that the set of $\mathbf{x} \in \mathbb{Z}^4_{\text{prim}}$ for which $[\mathbf{x}] \in S$ and $x_0x_1x_2x_3 = P_r$, is Zariski dense in S. The main aim of this paper is to show that $r(S) < \infty$ for a non-singular surface (the Fermat cubic surface) and a singular surface (the Cayley cubic surface).

Bourgain, Gamburd and Sarnak [2] and Nevo and Sarnak [17] established upper bounds for saturation numbers for orbits of congruence subgroups of semi-simple groups acting linearly on affine space. Moreover, Liu and Sarnak [15] considered the saturation number for certain affine quadric surfaces. However, these results do not cover the surfaces considered here.

The Fermat cubic surface is defined in $\mathbb{P}^3$ by the equation

$$S_1: \; x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0.$$

It is non-singular and has 27 lines, three of which are rational and take the form $x_i + x_j = x_k + x_l = 0$.

The Cayley cubic surface is defined in $\mathbb{P}^3$ by the equation

$$S_2: \; x_1x_2x_3 + x_0x_2x_3 + x_0x_1x_3 + x_0x_1x_2 = 0.$$

It has singularity type $4\mathbf{A}_1$. Moreover, there are 9 lines in the surface, three of which have the form $x_i + x_j = x_k + x_l = 0$, and the remaining six have the shape $x_i = x_j = 0$.

When considering the almost prime points on the surface, we are looking for the set of such points which form a Zariski dense subset. Thus it is not enough to work with almost prime points lying on individual curves contained in the surface. In particular, by writing U for the complement of the lines in the surface S, we may restrict attention to the open subset $U \subset S$.

To prove that a set of $[\mathbf{x}] \in S$ is Zariski dense, it suffices to prove that given $\varepsilon > 0$ and any $\boldsymbol{\xi} \in \mathbb{R}^4$ satisfying $[\boldsymbol{\xi}] \in U$, there exists $B \in \mathbb{N}$ sufficiently large and at least one point $[\mathbf{x}]$ in the set, such that

$$\left| \frac{\mathbf{x}}{B} - \boldsymbol{\xi} \right| < \varepsilon.$$

The following are our main results and establish the finiteness of $r(S_1)$ and $r(S_2)$.

Theorem 1. *Let $[\boldsymbol{\xi}] \in U_1(\mathbb{R})$ and $\varepsilon > 0$. Define*

$$M_{U_1}(\boldsymbol{\xi}, \varepsilon, B, r) = \# \left\{ \mathbf{x} \in \mathbb{Z}^4 : \begin{array}{c} [\mathbf{x}] \in U_1, \; \left| \frac{\mathbf{x}}{B} - \boldsymbol{\xi} \right| < \varepsilon, \\ x_0x_1x_2x_3 = P_r \end{array} \right\}.$$

Then for sufficiently large B, we have