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The representation of integers by positive ternary quadratic polynomials

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A B S T R A C T

An integral quadratic polynomial is called regular if it represents every integer that is represented by the polynomial itself over the reals and over the p-adic integers for every prime p. It is called complete if it is of the form $Q(\mathbf{x} + \mathbf{v})$, where Q is an integral quadratic form in the variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{v}$ is a vector in $\mathbb{Q}^n$. Its conductor is defined to be the smallest positive integer c such that $c\mathbf{v} \in \mathbb{Z}^n$. We prove that for a fixed positive integer c, there are only finitely many equivalence classes of positive primitive ternary regular complete quadratic polynomials with conductor c. This generalizes the analogous finiteness results for positive definite regular ternary quadratic forms by Watson [18,19] and for ternary triangular forms by Chan and Oh [8].

1. Introduction

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be an n-ary quadratic polynomial in variables $\mathbf{x} = (x_1, \dots, x_n)$ with rational coefficients. It takes the form

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$$f(\mathbf{x}) = Q(\mathbf{x}) + \ell(\mathbf{x}) + m$$

where $Q(\mathbf{x})$ is a quadratic form, $\ell(\mathbf{x})$ is a linear form, and m is a constant. Given a rational number a, it follows from the Hasse Principle that the diophantine equation

$$f(\mathbf{x}) = a \tag{1.1}$$

is soluble over the rationals if and only if it is soluble over the p-adic numbers for each prime p and over the reals. However, this local-to-global approach breaks down when we consider integral representations. Indeed, there are plenty of examples of quadratic polynomials for which (1.1) is soluble over each $\mathbb{Z}_p$ and over $\mathbb{R}$, but not soluble over $\mathbb{Z}$. Borrowing a term coined by Dickson for quadratic forms, we call a quadratic polynomial $f(\mathbf{x})$ *regular* if for every rational number a,

$$(1.1) \text{ is soluble over } \mathbb{Z} \Longleftrightarrow (1.1) \text{ is soluble over each } \mathbb{Z}_p \text{ and over } \mathbb{R}.$$

G.L. Watson [18,19] showed that there are only finitely many equivalence classes of primitive positive definite regular ternary quadratic forms. A list of representatives of these classes has been compiled in [11] by Jagy, Kaplansky, and Schiemann. Their list contains 913 ternary quadratic forms, and 891 are verified by them to be regular. Later B.-K. Oh [15] proves the regularity of 8 of the remaining 22 quadratic forms. More recently, R. Lemke Oliver [13] establishes the regularity of the last 14 quadratic forms under the generalized Riemann Hypothesis. Watson's result has been generalized by different authors to definite ternary quadratic forms over other rings of arithmetic interest [3,5], to higher dimensional representations of positive definite quadratic forms in more variables [6], and to positive definite ternary quadratic forms which satisfy other regularity conditions [4].

There are regular quadratic polynomials that are not quadratic forms. A well-known example is the sum of three triangular numbers

$$\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \frac{x_3(x_3+1)}{2},$$

which is universal (i.e. representing all positive integers) and hence regular. Given positive integers a, b, c, we follow the terminology in [7] and call the polynomial

$$\Delta(a,b,c) := a\frac{x_1(x_1+1)}{2} + b\frac{x_2(x_2+1)}{2} + c\frac{x_3(x_3+1)}{2}$$

a triangular form. It is *primitive* if $\gcd(a,b,c) = 1$. There are seven universal ternary triangular forms –hence all are regular– and they were found by Liouville in 1863 [14]. An example of a regular ternary triangular form which is not universal is $\Delta(1,1,3)$. We offer a proof in the following example.