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The number of rational points of a family of hypersurfaces over finite fields

Shuangnian Hu [a,b], Shaofang Hong [a,*,1], Wei Zhao [a,c]

[a] Mathematical College, Sichuan University, Chengdu 610064, PR China
[b] School of Mathematics and Physics, Nanyang Institute of Technology, Nanyang 473004, PR China
[c] Science and Technology on Communication Security Laboratory, Chengdu 610041, PR China

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ABSTRACT

Let $\mathbb{F}_q$ denote the finite field of odd characteristic p with q elements ($q = p^e$, $e \in \mathbb{N}$) and $\mathbb{F}_q^*$ represent the nonzero elements of $\mathbb{F}_q$. In this paper, by using the Smith normal form of the index matrix, we give a formula for the number of rational points of the following family of hypersurface over $\mathbb{F}_q$:

$$\sum_{j=0}^{t-1}\sum_{i=1}^{r_{j+1}-r_j} a_{r_j+i} x_1^{e_{r_j+i,1}} \dots x_{n_{j+1}}^{e_{r_j+i,n_{j+1}}} - b = 0,$$

where the integers $t > 0$, $r_0 = 0 < r_1 < r_2 < \ldots < r_t$, $n_1 < n_2 < \ldots < n_t$, $b \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^*$ ($i = 1, \ldots, r_t$), and the index of each variable is a positive integer. Especially under some certain conditions, we get an explicit formula of the number of rational points of the above hypersurface. This generalizes greatly the results obtained by Wolfmann in 1994, Sun in 1997 and Wang and Sun in 2005, respectively.

* Corresponding author.
E-mail addresses: hushuangnian@163.com (S. Hu), sfhong@scu.edu.cn, s-f.hong@tom.com, hongsf02@yahoo.com (S. Hong), zhaowei9801@163.com (W. Zhao).

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1. Introduction

Let $\mathbb{F}_q$ denote the finite field of q elements with odd characteristic p ($q = p^e$, $e \in \mathbb{N}$ (the set of positive integers)) and $\mathbb{F}_q^*$ represent the nonzero elements of $\mathbb{F}_q$. Let $f(x_1, \ldots, x_n)$ be a polynomial with n variables over $\mathbb{F}_q$ and $N(f = 0)$ denote the number of $\mathbb{F}_q$-rational points on the affine hypersurface $f = 0$ in $\mathbb{F}_q^n$. That is,

$$N(f = 0) = \#\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n | f(x_1, \ldots, x_n) = 0\}.$$

Studying the value of $N(f = 0)$ is one of the main topics in finite fields. Generally speaking, it is nontrivial to give the formula for $N(f = 0)$. The $\deg(f) = d$ plays an important role in the estimate of $N(f = 0)$. An upper bound for $N(f = 0)$ is given in [16] by

$$N(f = 0) \leq dq^{n-1}.$$

We use ord_p to denote the p-adic additive valuation normalized so that $\mathrm{ord}_p p = 1$. The classical Chevalley–Warning theorem shows that $\mathrm{ord}_p N(f = 0) > 0$ if $n > d$. Let $\lceil x \rceil$ denote the least integer $\geq x$ and e denote the extension degree of $\mathbb{F}_q/\mathbb{F}_p$. Ax [1] showed that

$$\mathrm{ord}_p N(f = 0) \geq e\left\lceil \frac{n-d}{d} \right\rceil.$$

The Chevalley–Warning–Ax-type estimates can be improved in many special cases (see [2,3,5,10,11,13,15,17,23–25]).

Finding the explicit formula for $N(f = 0)$ under certain condition has attracted lots of authors for many years. From [14] and [16] we know that there exists an explicit formula for $N(f = 0)$ satisfying $\deg(f) \leq 2$ in $\mathbb{F}_q$. For the following diagonal hypersurface

$$a_1 x_1^{e_1} + \ldots + a_n x_n^{e_n} - b = 0,\ 1 \leq i \leq n,\ a_i \in \mathbb{F}_q^*,\ b \in \mathbb{F}_q,\ e_i > 0, \tag{1.1}$$

much work has been done to seek for the number of rational points of the hypersurface (1.1), see, for instance, [21,22,27]. Regarding the following k-linear hypersurface

$$a_1 x_{11} \ldots x_{1k} + a_2 x_{21} \ldots x_{2k} + \ldots + a_n x_{n1} \ldots x_{nk} - b = 0, \tag{1.2}$$

with $a_i \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$, Carlitz [6], Cohen [9], and Hodges [10] also obtained the formula of the number of its rational points independently. Especially, Cao and Sun [4] studied the general diagonal hypersurface

$$a_1 x_{11}^{e_{11}} \ldots x_{1n_1}^{e_{1n_1}} + a_2 x_{21}^{e_{21}} \ldots x_{2n_2}^{e_{2n_2}} + \ldots + a_r x_{r1}^{e_{r1}} \ldots x_{rn_r}^{e_{rn_r}} = 0,$$

with $1 \leq i \leq r$, $1 \leq j \leq n_i$, $e_{ij} \in \mathbb{N}$, $a_i \in \mathbb{F}_q^*$. Clearly, this extends (1.1) and (1.2) when $b = 0$. Recently, Pan, Zhao and Cao [18] presented a formula of the number of rational