Journal of Number Theory 156 (2015) 154–160

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Proof of a conjecture of Z.-W. Sun on the divisibility of a triple sum

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ARTICLE INFO

Article history:
Received 27 January 2015
Received in revised form 21 April 2015
Accepted 21 April 2015
Available online 6 June 2015
Communicated by David Goss

MSC:
11A07
11B65
05A10

Keywords:
Congruence
Supercongruence
Zeilberger's algorithm

ABSTRACT

Define the numbers R_n and W_n by

$$R_n = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1}, \text{ and}$$

$$W_n = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{3}{2k-3}.$$

We prove that, for any positive integer n and odd prime p, there hold

$$\sum_{k=0}^{n-1}(2k+1)R_k^2 \equiv 0 \pmod{n},$$

$$\sum_{k=0}^{p-1}(2k+1)R_k^2 \equiv 4p(-1)^{\frac{p-1}{2}} - p^2 \pmod{p^3},$$

$$9\sum_{k=0}^{n-1}(2k+1)W_k^2 \equiv 0 \pmod{n},$$

$$\sum_{k=0}^{p-1}(2k+1)W_k^2 \equiv 12p(-1)^{\frac{p-1}{2}} - 17p^2 \pmod{p^3}, \quad \text{if } p>3.$$

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http://dx.doi.org/10.1016/j.jnt.2015.04.024

The first two congruences were originally conjectured by Z.-W. Sun.

1. Introduction

It is easy to see that $\binom{2n}{n}\frac{1}{2n-1}$ is always an integer for $n \geqslant 0$. Recently, Z.-W. Sun [7] introduced the following numbers

$$R_n = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1},$$

and proved some interesting arithmetic properties of these numbers. For example, Sun [7] proved that, if p is a prime of the form $4k+1$, then $R_{\frac{p-1}{2}} \equiv p-(-1)^{\frac{p-1}{4}}2x \pmod{p^2}$, where $p = x^2+y^2$ with $x \equiv 1 \pmod 4$.

The first aim of this paper is to prove the following result, which was originally conjectured by Z.-W. Sun (see [7, Conjecture 5.4]).

Theorem 1.1. *Let n be a positive integer and p an odd prime. Then*

$$\sum_{k=0}^{n-1}(2k+1)R_k^2 \equiv 0 \pmod{n}, \tag{1.1}$$

$$\sum_{k=0}^{p-1}(2k+1)R_k^2 \equiv 4p(-1)^{\frac{p-1}{2}} - p^2 \pmod{p^3}. \tag{1.2}$$

Since $\binom{2n}{n}\frac{3}{2n-3} = \binom{2n}{n}\frac{1}{2n-1} + \binom{2n-2}{n-1}\frac{8}{2n-3}$, we see that $\binom{2n}{n}\frac{3}{2n-3}$ is always an integer. Let

$$W_n = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{3}{2k-3}.$$

The second aim of this paper is to prove the following result.

Theorem 1.2. *Let n be a positive integer and let $p > 3$ be a prime. Then*

$$9\sum_{k=0}^{n-1}(2k+1)W_k^2 \equiv 0 \pmod{n}, \tag{1.3}$$

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