Journal of Number Theory 156 (2015) 161–182

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the quantitative dynamical Mordell–Lang conjecture

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ARTICLE INFO

Article history:
Received 25 March 2015
Received in revised form 19 April 2015
Accepted 21 April 2015
Available online 6 June 2015
Communicated by Kenneth A. Ribet

MSC:
primary 37P55
secondary 11B37, 11D61, 11D72

Keywords:
Dynamical Mordell–Lang conjecture
Linear recurrence sequence
Exponential polynomial
Linear equation

ABSTRACT

The dynamical Mordell–Lang conjecture concerns the structure of the intersection of an orbit in an algebraic dynamical system and an algebraic variety. In this paper, we bound the size of this intersection for various cases when it is finite.

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http://dx.doi.org/10.1016/j.jnt.2015.04.011
0022-314X/

1. Introduction

1.1. Motivation

Let $\mathcal{X}$ be an algebraic variety defined over the complex numbers $\mathbb{C}$, and let $\Phi : \mathcal{X} \to \mathcal{X}$ be a morphism. For any integer $n \geq 0$, we denote by $\Phi^{(n)}$ the n-th iteration of Φ with $\Phi^{(0)}$ denoting the identity map.

Throughout the paper, a single integer is viewed as an arithmetic progression with common difference 0.

The following is the well-known *dynamical Mordell–Lang conjecture* for self-morphisms of algebraic varieties in the dynamical setting; see [11,16,17].

Conjecture 1.1 *(Dynamical Mordell–Lang Conjecture). Let $\mathcal{X}$ and Φ be given as the above, let $V \subseteq \mathcal{X}$ be a closed subvariety, and let $P \in \mathcal{X}(\mathbb{C})$. Then, the following subset of integers*

$$\{n \geq 0 : \Phi^{(n)}(P) \in V(\mathbb{C})\}$$

is a finite union of arithmetic progressions.

Conjecture 1.1 has been studied extensively in recent years. However, so far there are only a few related results. These include results on maps of various special types [4,5,7,14,16,17,23,24] (especially diagonal maps), and analogues for Noetherian spaces [6] and Drinfeld modules [15].

Recently, Silverman and Viray [23, Corollary 1.4] have given results regarding the uniform boundedness (only in terms of m) of intersections of orbits of the power map (with the same exponent) at a point of the projective m-space $\mathbb{P}^m(\mathbb{C})$ with non-zero multiplicatively independent coordinates, with any linear subspace of $\mathbb{P}^m(\mathbb{C})$. However, they have not provided quantitative results. In fact, such a result follows, even in a more general case, directly from the uniform bound on the number of zeros of simple and non-degenerate linear recurrence sequences.

We also note that the uniform boundedness condition has recently been considered in [10], where several results are given for the frequency of the points in an orbit of an algebraic dynamical system that belong to a given algebraic variety under the reduction modulo a prime p.

1.2. Our results

In this paper, we study the quantitative version of Conjecture 1.1 for polynomial morphisms of several special types when $\mathcal{X}$ is the affine m-space $\mathbb{A}^m(\mathbb{C})$ and V is a hypersurface; see Section 3. Our main objective is to find as many classes of polynomial morphisms as possible having uniform bounds (or as close as possible to uniformity), and not to investigate detailedly the quality of the bounds. To the best of our knowledge, this is the first work on the quantitative dynamical Mordell–Lang conjecture.