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A characterization of almost universal ternary inhomogeneous quadratic polynomials with conductor 2

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ABSTRACT

An integral quadratic polynomial (with positive definite quadratic part) is called almost universal if it represents all but finitely many positive integers. In this paper, we provide a characterization of almost universal ternary quadratic polynomials with conductor 2.

1. Introduction

For a polynomial $f(x_1, \ldots, x_n)$ with rational coefficients and an integer a, we say that f represents a if the Diophantine equation $f(x_1, \ldots, x_n) = a$ has a solution over the integers. One particularly interesting question asks, when is a polynomial *almost universal*;

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that is, when does a polynomial represent all but finitely many natural numbers? In the specific case when f is a quadratic form, this question has attracted a great deal of interest over the last years.

Given a quadratic map Q and $\mathbb{Z}^n$ in the standard basis, the pair $(\mathbb{Z}^n, Q)$ is a $\mathbb{Z}$-lattice of rank n, which we denote by N. Since the representation behavior for indefinite lattices is well understood, all lattices in this paper are assumed to be positive definite; for the indefinite case the reader is referred to the survey paper by Hsia [10].

A homogeneous integral quadratic polynomial can always be viewed as a quadratic lattice. For rank greater than 4, Tartakowsky's results in [16] imply that a lattice is almost universal if it is universal over $\mathbb{Z}_p$ for every prime p. In the quaternary case, Bochnak and Oh [1] give an effective method to determine when a lattice is almost universal, resolving an investigation first initiated by Ramanujan in [14]. For the ternary case, it is a well known consequence of Hilbert Reciprocity that a positive definite ternary $\mathbb{Z}$-lattice is anisotropic at an odd number of finite primes, and therefore it is not universal at these primes. Hence the lattice fails to represent an entire square class in $\mathbb{Q}_p/\mathbb{Q}_p^\times$ at these primes, and hence cannot be almost universal.

Therefore, we turn out attention to inhomogeneous quadratic polynomials of the form

$$f(x) = Q(x) + L(x) + c,$$

where $Q(x)$ is a quadratic form, L is a linear form, and c is a constant. It is not a surprise that we can study the arithmetic of these polynomials from the geometric perspective of quadratic spaces and lattices. Indeed, Q can be viewed as the quadratic map on $N = (\mathbb{Z}^n, Q)$, and associated with Q is the symmetric bilinear map B. Under the assumption that Q is positive definite, $L(x) = 2B(\nu, x)$ for a unique choice of vector ν in $\mathbb{Q}N$ which is the quadratic space underlying N. The choices for ν and N are completely determined by the coefficients of Q and L. Since the constant c does not contribute anything essential to the arithmetic of f, there is no harm in assuming that it is equal to zero. Thus, an integer a is represented by $f(x)$ if and only if $Q(\nu) + a$ is represented by the coset $\nu + N$. In general, there is no local–global principle for representations of integers by cosets of quadratic lattices. However, when $n \geq 4$, Chan and Oh [3, Theorem 4.9] show how the asymptotic local–global principles for representations with approximation property by Jöchner–Kitaoka [12] and by Hsia–Jöchner [11] lead to an asymptotic local–global principle for representations of integers by cosets. Therefore we restrict the discussions of this paper to the ternary case.

Given a lattice N and a vector $\nu \in \mathbb{Q}N$, we define the conductor $\mathfrak{m}$ of $\nu + N$ as in [8]; that is, the minimal integer for which $\mathfrak{m}\nu \in N$. In [8] we give a characterization of almost universal ternary inhomogeneous quadratic polynomials where $\mathfrak{m}$ is an odd prime power.