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Large and moderate deviation principles for alternating Engel expansions

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A B S T R A C T

In this paper, we investigate the large and moderate deviation principles for alternating Engel expansions, a classical representation of real numbers in number theory.

1. Introduction

For any real number $x \in (0, 1]$ and $n \geq 1$, we define

$$x = x_1, \qquad d_n := d_n(x) = \left\lfloor \frac{1}{x_n} \right\rfloor \quad \text{and} \quad x_{n+1} = 1 - x_n d_n, \tag{1.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Then for every $x \in (0, 1]$, the algorithm (1.1) uniquely generates a finite or infinite series. That is,

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$$x = \frac{1}{d_1} - \frac{1}{d_1 d_2} + \cdots + (-1)^{n-1} \frac{1}{d_1 d_2 \cdots d_n} + \cdots, \tag{1.2}$$

where $d_n \geq 1$ are positive integers with $d_{n+1} \geq d_n + 1$ for all $n \geq 1$. The representation (1.2) is said to be the *alternating Engel expansion (or, Pierce expansion)* of x and $d_n(x)$, $n \geq 1$ are called the *digits* of the alternating Engel expansion of x. We sometimes write the form (1.2) as $x = ((d_1, d_2, \cdots, d_n, \cdots))$. This expansion was first considered by Pierce [10] in 1929. Furthermore, some arithmetic and statistical properties of the alternating Engel expansion, such as the representation of rational numbers, law of large numbers, central limit theorem and law of the iterated logarithm were studied by Remez [11], Shallit [12] and Valēēv and Zlēbov [15]. Later, Shallit [13] applied this expansion to proposing a very nice method for determining leap years which generalizes those existent in 1994. For more details about the alternating Engel expansion, we refer the reader to [3,4,6,17] and the references therein.

Now we turn to introducing the large and moderate deviation principles. Let $\{X_n : n \geq 1\}$ be a sequence of real-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A function $I : \mathbb{R} \to [0, \infty]$ is called a *good rate function* if it is lower semicontinuous and has compact level sets. Let $\{\lambda_n : n \geq 1\}$ be a sequence of positive real numbers with $\lim\limits_{n\to\infty} \lambda_n = \infty$. We say that the sequence $\{X_n : n \geq 1\}$ satisfies a *large deviation principle* (LDP for short) with speed λ_n and good rate function I under $\mathbf{P}$, if for any Borel set Γ,

$$-\inf_{x\in\Gamma^\circ} I(x) \leq \liminf_{n\to\infty} \frac{1}{\lambda_n} \log \mathbf{P}(X_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \mathbf{P}(X_n \in \Gamma) \leq -\sup_{x\in\bar{\Gamma}} I(x),$$

where Γ° and $\bar{\Gamma}$ denotes the interior and the closure of Γ respectively. Formally, there is no distinction between the large deviation principle and the moderate deviation principle (MDP for short). Usually LDP characterizes the convergence speed of the law of large numbers, while MDP describes the speed of convergence between the law of large numbers and the central limit theorem. For an introduction to the theory of large and moderate deviation principles, we refer the reader to Dembo and Zeitouni [1], Touchette [14] and Varadhan [16].

We here denote by $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space, where $\Omega = (0, 1]$, $\mathcal{F}$ is the Borel σ-algebra on $(0, 1]$ and $\mathbf{P}$ denotes the Lebesgue measure on $(0, 1]$. In [12], Shallit established a relation between Stirling numbers of the first kind (see Jordan [7]) and the distribution of the digit d_n occurring in the alternating Engel expansion to estimate the expectation and variance of quantities connected with d_n. Using these estimates, Shallit showed a strong law of large numbers for the digit sequence $\{d_n : n \geq 1\}$, i.e., for $\mathbf{P}$-almost all $x \in (0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log d_n(x) = 1.$$