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Uniform positive existential interpretation of the integers in rings of entire functions of positive characteristic

Natalia Garcia-Fritz [a,1], Hector Pasten [b,*,2]

[a] *Department of Mathematics and Statistics, Queen's University, Jeffery Hall, University ave., Kingston, ON K7L 3N6, Canada*
[b] *Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA*

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A B S T R A C T

We prove a negative solution to the analogue of Hilbert's tenth problem for rings of one variable non-Archimedean entire functions in any characteristic. In the positive characteristic case we prove more: the ring of rational integers is uniformly positive existentially interpretable in the class of $\{0, 1, t, +, \cdot, =\}$-structures consisting of positive characteristic rings of entire functions on the variable t. From this we deduce uniform undecidability results for the positive existential theory of such structures.

* Corresponding author.
E-mail addresses: ngarciafritz@gmail.com (N. Garcia-Fritz), hpasten@gmail.com (H. Pasten).

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1. Introduction and results

In this work we establish a negative solution to the analogue of Hilbert's tenth problem for rings of one variable non-Archimedean entire functions in any characteristic (such a result for one variable non-Archimedean entire functions was only known in characteristic zero, see [11]). Moreover, we prove that the ring $\mathbb{Z}$ is *uniformly* positive existentially interpretable in the class of rings of one variable entire functions in positive characteristic, which leads to uniform undecidability results in the positive characteristic case (extending some results from [15]). Let us introduce some notation before stating our results in a precise way.

Let R be an integral domain endowed with an absolute value $|\cdot|$. We denote by $\mathcal{A}_R$ the ring of entire functions defined over R, by which we mean the ring of power series in the variable t with coefficients in R and infinite radius of convergence. More precisely, when $|\cdot|$ is Archimedean this ring is defined as

$$\mathcal{A}_R = \left\{ \sum_{n\geq 0} c_n t^n : \forall r \in \mathbb{R}^+, \lim_{M\to\infty} \sum_{n\geq M} |c_n| r^n = 0 \right\}$$

while when $|\cdot|$ is non-Archimedean this ring is defined as

$$\mathcal{A}_R = \left\{ \sum_{n\geq 0} c_n t^n : \forall r \in \mathbb{R}^+, \lim_{M\to\infty} |c_M| r^M = 0 \right\}.$$

We remark that the elements of $\mathcal{A}_R$ define (by evaluation) functions $R \to R$ when R is complete, but not in general. Also, there is a slight abuse of notation (the absolute value is not explicit in $\mathcal{A}_R$) but this should not lead to any confusion in the present work.

For different choices of R one recovers some familiar examples of $\mathcal{A}_R$. For instance, $\mathcal{A}_\mathbb{C}$ is the ring $\mathcal{H}$ of holomorphic functions on $\mathbb{C}$, while $\mathcal{A}_{\mathbb{C}_p}$ is the ring of analytic functions $\mathcal{H}_p$ on $\mathbb{C}_p$ (as usual, $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$). Also, observe that any ring R can be given the trivial absolute value, in which case $\mathcal{A}_R = R[t]$. When the absolute value is non-trivial, $\mathcal{A}_R$ can strictly contain $R[t]$ even in positive characteristic (which is the most relevant case for our purposes); for example, we can take $R = \mathbb{F}_p[[q]]$ (q is transcendental over $\mathbb{F}_p$) with the valuation of the order at q, then the power series

$$\sum_{n=1}^{\infty} q^{n^2} t^n \in R[[t]]$$

is an element of $\mathcal{A}_R$ and it is not in $R[t]$.

The arithmetic of $\mathcal{A}_R$ shares many similarities with the arithmetic of $\mathbb{Z}$. This analogy is classically known in the case of $\mathcal{A}_{\mathbb{F}_q} = \mathbb{F}_q[t]$. For $R = \mathbb{C}$, and for non-Archimedean (complete, algebraically closed) fields R in any characteristic with non-trivial absolute value,