Journal of Number Theory 158 (2016) 23–32

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Cooper and Lam's conjecture for generalized Bell ternary quadratic forms

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ARTICLE INFO

Article history:
Received 20 April 2015
Received in revised form 2 June 2015
Accepted 2 June 2015
Available online 6 August 2015
Communicated by David Goss

MSC:
11E25
11D45

Keywords:
Sum of squares
Ternary quadratic form
Theta function
Hurwitz three-squares formula
Cooper and Lam's conjecture

ABSTRACT

Bell's theorem determines the counting function of the ternary quadratic forms $x^2 + by^2 + cz^2$, with $b, c \in \{1, 2, 4, 8\}$, in terms of the number $r_3(n)$ of representations of n as a sum of three squares. Based on it we verify Cooper and Lam's conjecture for them. This result includes two new cases so far left open. Additionally, we show that the forms $(b, c) = (2, 16)$ and $(b, c) = (8, 16)$ are generalized Bell forms in the sense that their counting functions depend only upon $r_3(n)$. These forms satisfy Cooper and Lam's conjecture and solve two further open cases.

1. Introduction

Given a ternary diagonal quadratic form $Q(x, y, z) = ax^2 + by^2 + cz^2$, one is in general interested in the total number $r_Q(n)$ of integer solutions of the Diophantine equation $Q(x, y, z) = n$. This number is denoted by $r_{(a,b,c)}(n)$ while $r_{(1,1,1)}(n)$ is usually abbreviated by $r_3(n)$. A formula for the number $r_3(n^2)$ of representations of n^2 by a sum

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http://dx.doi.org/10.1016/j.jnt.2015.06.020

of three squares is due to Hurwitz [12]:

$$r_3(n^2) = 6 \cdot S(m), \quad \text{with } S(m) = \prod_{p|m} \left\{ \sigma(p^{\lambda_p}) - \left(\frac{-1}{p}\right) \sigma(p^{\lambda_p - 1}) \right\}, \tag{1.1}$$

where $\sigma(m)$ counts the sum of all positive divisors of m, $(\frac{-1}{p})$ is the Legendre symbol, and $n = 2^{\lambda_2} \cdot m$, $m = \prod_{p|m} p^{\lambda_p}$, is the unique decomposition of n in prime numbers p (see Dickson [6], p. 271). An elementary proof of it is due to Lagrange [13].

Cooper and Lam [5] found a number of formulas similar to (1.1) for the ternary quadratic forms $(1, 1, 2)$, $(1, 2, 2)$, $(1, 1, 3)$ and $(1, 3, 3)$. Based on computer investigations over the domain $1 \leq b \leq c \leq 200$ these authors raised a conjecture for a set of 69 ternary quadratic forms of the type $(1, b, c)$. The counting functions of these forms are claimed to satisfy a formula of the type

$$r_{(1,b,c)}(n^2) = \left(\prod_{p|2bc} g(b, c, p, \lambda_p) \right) \left(\prod_{p \in P} \left\{ \sigma(p^{\lambda_p}) - \left(\frac{-bc}{p}\right) \sigma(p^{\lambda_p - 1}) \right\} \right), \tag{1.2}$$

where P consists of the number of primes $p|n$ that do not divide $2bc$, and $g(b, c, p, \lambda_p)$ has to be determined on an individual and case-by-case basis. This contemporary conjecture has been further tackled by Guo et al. [7] and Ye [19] but remains unsolved in general.

Interestingly enough one observes that the 69 conjectured forms contain the 10 different ternary quadratic forms of the type $(1, b, c)$ with $b, c \in \{1, 2, 4, 8\}$ studied first by Bell [1]. Using 13 identities about theta functions, including some important ones by Kronecker and Hermite, Bell has determined $r_{(1,b,c)}(n)$ for these forms, which depend only on $r_3(n)$. A modern elementary and simpler proof of Bell's theorem is contained in Hürlimann [9]. In view of this result we call a ternary quadratic form a *generalized Bell form* in case its counting function depends only on $r_3(n)$. We ask whether the original Bell and generalized Bell ternary quadratic forms satisfy Cooper and Lam's conjecture.

A brief description of the content follows. Section 2 begins with Bell's theorem. Then, Cooper and Lam's conjecture is proved for these ternary quadratic forms. Besides the elementary derivation, that in some cases directly follows from a remark by Bell [1], this result includes two new so far open cases. Section 3 shows that the ternary quadratic forms $(1, 2, 16)$ and $(1, 8, 16)$ are generalized Bell forms and proves Cooper and Lam's conjecture for them.

2. Cooper and Lam's conjecture for Bell's ternary quadratic forms

We begin with Bell's theorem.

Theorem 2.1 *(Theorem of Bell). The nine counting functions $r_{(1,b,c)}(n)$, $b, c \in \{1, 2, 4, 8\}$, $b \leq c$, $(b, c) \neq (1, 1)$, are determined as follows:*