Journal of Number Theory 158 (2016) 104–133

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Points at rational distances from the vertices of certain geometric objects

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ARTICLE INFO

Article history:
Received 28 December 2014
Received in revised form 25 May 2015
Accepted 3 June 2015
Available online 6 August 2015
Communicated by David Goss

MSC:
11D25
14J20
11G35
14G05

Keywords:
Rational points
Elliptic surfaces
Rational distances set

ABSTRACT

We consider various problems related to finding points in $\mathbb{Q}^2$ and in $\mathbb{Q}^3$ which lie at rational distance from the vertices of some specified geometric object, for example, a square or rectangle in $\mathbb{Q}^2$, and a cube or tetrahedron in $\mathbb{Q}^3$. In particular, as one of several results, we prove that the set of positive rational numbers a such that there exist infinitely many rational points in the plane which lie at rational distance from the four vertices of the rectangle with vertices $(0, 0)$, $(0, 1)$, $(a, 0)$, and $(a, 1)$, is dense in $\mathbb{R}_+$.

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[1] The research of the second author was supported by the grant of the Polish National Science Centre no. UMO-2012/07/E/ST1/00185.

http://dx.doi.org/10.1016/j.jnt.2015.06.011
0022-314X/

1. Introduction

Berry [1] showed that the set of rational points in the plane with rational distances to three given vertices of the unit square is infinite. More precisely, he showed that the set of rational parametric solutions of the corresponding system of equations is infinite; this generalizes some earlier work of Leech. In a related work, he was able to show that for any given triangle ABC in which the length of at least one side is rational and the squares of the lengths of all sides are rational, then the set of points P with rational distances $|PA|$, $|PB|$, $|PC|$ to the vertices of the triangle is dense in the plane of the triangle; see Berry [2]. However, it is a notorious and unsolved problem to determine whether there exists a rational point in the plane at rational distance from the *four* corners of the unit square (see Problem D19 in Guy's book [7]). Such a point corresponds to a rational solution of a Diophantine system comprising four quadratics in six-dimensional projective space. An analogous system is that for the "rational cuboid" problem, of finding a rectangular cuboid with rational edges, face diagonals, and body diagonal. This again is a notorious and unsolved problem. A great deal of computing has produced no solution, and it is tempting to believe that no solution exists. However, another naturally arising Diophantine system of four quadratics in six-dimensional projective space is that of finding a three-by-three magic square of integers in which a certain seven of the entries are perfect squares. This system *does* have a solution (although only one solution up to symmetries is known); see Bremner [3]. Because of the difficulty of this problem one can ask a slightly different question, as to whether there exist rational points in the plane which lie at rational distance from the four vertices of the *rectangle* with vertices $(0, 0)$, $(0, 1)$, $(a, 0)$, and $(a, 1)$, for $a \in \mathbb{Q}$. This problem is briefly alluded to in Section D19 on p. 284 of Guy's book. In Section 2 we reduce this problem to the investigation of the existence of rational points on members of a certain family of algebraic curves $\mathcal{C}_{a,t}$ (depending on rational parameters a, t). We show that the set of $a \in \mathbb{Q}$ for which the set of rational points on $\mathcal{C}_{a,t}$ is infinite is dense in $\mathbb{R}$ (in the Euclidean topology).

Richard Guy has pointed out that there are immediate solutions to the four-distance unit square problem if the point is allowed to lie in three space $\mathbb{Q}^3$. Indeed, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$ lies at rational distance to the four vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ of the square. This observation leads us to consider the more general problem, of points in $\mathbb{Q}^3$ which lie at rational distance from the four vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ of the unit square. In Section 3 we show that such points are dense on the line $x = \frac{1}{2}$, $y = \frac{1}{2}$, and dense on the plane $x = \frac{1}{2}$. Further, there are infinitely many parameterizations of such points on the plane $x = y$. In Section 4 we consider the general problem of finding points $(x, y, z) \in \mathbb{Q}^3$ with rational distances to the vertices of a unit square lying in the plane $z = 0$ without any assumptions on x, y, z. Attempts to show such points are dense in $\mathbb{R}^3$ have been unsuccessful to date. However, we are able to show that the variety related to this problem is unirational over $\mathbb{Q}$. In particular, this implies the existence of a parametric family of rational points with rational distances to the four vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 0)$ of the unit square. Whether there exist points in $\mathbb{Q}^3$ at rational