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A refinement of the Dress–Scharlau theorem

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ABSTRACT

In 1982, Dress and Scharlau [1] found an upper bound for the norm of totally positive, additively indecomposable algebraic integers in real quadratic fields and showed that this bound is sharp if the norm of the fundamental unit is −1. In this paper, we prove that their upper bound is not extremal if the norm of the fundamental unit is +1 and establish a new upper bound that is sharp for a large family of such fields.

1. Introduction

A totally positive algebraic integer of a totally real number field K is additively indecomposable if it cannot be represented by the sum of two totally positive integers over K. Since there are finitely many additively indecomposable algebraic integers over K up to multiplication by a unit, there is a maximum N_K of the norm of these integers. Let $K = K_d$ be a real quadratic field $\mathbb{Q}(\sqrt{d})$. An upper bound of N_{K_d} is found by Dress and Scharlau [1] in 1982. They showed if $d \equiv 2, 3 \pmod 4$, then $N_{K_d} \leq d$, and if $d \equiv 1 \pmod 4$, then $N_{K_d} \leq \frac{d-1}{4}$. They also showed the equality holds if the norm of

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the fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$ is -1. Recently the first author of this paper found a generalization of the Dress–Scharlau Theorem. He showed [2] if c_d is the upper bound obtained by Dress and Scharlau, then the norm of totally positive algebraic integers over K_d which cannot be represented by the sum of $n+1$ totally positive integers over K is less than or equal to n^2c_d if $d \equiv 2, 3 \pmod 4$, $n^2c_d + \frac{n^2}{4}$ if $d \equiv 1 \pmod 4$ and n is even, $n^2c_d + \frac{n^2-1}{4}$ if $d \equiv 1 \pmod 4$ and n is odd. The equality also holds if the norm of ε_d is -1.

This paper treats the norm of totally positive algebraic integers over $K_d = \mathbb{Q}(\sqrt{d})$ when the norm of ε_d is 1. Let $-k$ be the maximum negative norm of the algebraic integers over $\mathbb{Q}(\sqrt{d})$. In other words k is the smallest positive rational integer such that there is an algebraic integer α over $\mathbb{Q}(\sqrt{d})$ whose norm is $-k$. We will show that

$$N_{K_d} \leq \frac{c_d}{k} = \begin{cases} \frac{d}{k}, & \text{when } d \equiv 2, 3 \pmod 4 \text{ ;} \\ \frac{d-1}{4k}, & \text{when } d \equiv 1 \pmod 4. \end{cases}$$

The equality holds if d is a multiple of k. Moreover there is an additively indecomposable totally positive algebraic integer over K_d whose norm is at least $\frac{c_d}{k} - \frac{k}{4}$. We also compute N_{K_d} for some families of $K_d = \mathbb{Q}(\sqrt{d})$.

2. The basic definitions and theorems

For a non-square positive integer d, $K_d = \mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in K_d\}$ is a real quadratic field. For $\alpha = a + b\sqrt{d} \in K_d$, $\bar{\alpha} = a - b\sqrt{d}$ is the *conjugate* of α. The *trace* of α, denoted by $\mathrm{tr}(\alpha)$, is defined $\mathrm{tr}(\alpha) = \alpha + \bar{\alpha}$. The *norm* of α, denoted by $N(\alpha)$, is defined by $N(\alpha) = \alpha\bar{\alpha}$. An element $\alpha \in K_d$ is an *algebraic integer* of $\mathbb{Q}(\sqrt{d})$ if and only if $\alpha + \bar{\alpha}, \alpha\bar{\alpha} \in \mathbb{Z}$. An element α of K_d is *totally positive* if and only if $\alpha > 0$ and $\bar{\alpha} > 0$. For a non-square positive integer d, let

$$\omega_d = \begin{cases} \sqrt{d}, & d \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{d}}{2}, & d \equiv 1 \pmod 4 \end{cases}.$$

The set $\mathcal{O}_d$ is the ring of all algebraic integers of $\mathbb{Q}(\sqrt{d})$. Then $\mathcal{O}_d = \{a + b\omega_d \mid a, b \in \mathbb{Z}\}$ is the ring of all algebraic integers over K_d, $\mathcal{O}_d^+ = \{\alpha \in \mathcal{O}_d | \alpha > 0, \overline{\alpha} > 0\}$ is the set of all totally positive integers over K_d and $\mathcal{O}_d^- = \{\alpha \in \mathcal{O}_d | \alpha < 0, \overline{\alpha} < 0\}$ is the set of all totally negative integers over K_d. We define $\mathcal{O}_d^\pm = \{\alpha \in \mathcal{O}_d | \alpha > 0, \overline{\alpha} < 0\}$ is the set of all *partially positive* integers over K_d, $\mathcal{O}_d^\mp = \{\alpha \in \mathcal{O}_d | \alpha < 0, \overline{\alpha} > 0\}$ is the set of all *partially negative* integers over K_d.

Let σ be one of $+$, $\pm$, $\mp$, $-$. Then $\alpha \in \mathcal{O}_d^\sigma$ is additively indecomposable in $\mathcal{O}_d^\sigma$ if and only if there are no $\beta, \gamma \in \mathcal{O}_d^\sigma$ such that $\alpha = \beta + \gamma$.

For $d > 0$, there is a unit $\varepsilon = \varepsilon_d$, called the fundamental unit of $\mathbb{Q}(\sqrt{d})$ such that $\varepsilon > 1$ and the group U_d of units in $\mathcal{O}_d$ is $\{\pm\varepsilon_d^n \mid n \in \mathbb{Z}\}$.