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On the p-rank of tame kernel of number fields ☆

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ABSTRACT

In this paper, the relations between p-ranks of the tame kernel and the ideal class group for a general number field are investigated. As a result, nearly all of Browkin's results about quadratic fields are generalized to those for general number fields. In particular, a p-rank formula between the tame kernel and the ideal class group for a totally real number field of odd degree is obtained when p is a Fermat prime. As an example, the case of cyclic quartic fields is considered in more details. More precisely, using the results on cyclic quartic fields, we give some results connecting the $p\text{-rank}(K_2\mathcal{O}_F)$ with the $p\text{-rank}(\text{Cl}(\mathcal{O}_E))$, where F is a cyclic quartic field and E is an appropriate subfield of $F(\zeta_p)$.

1. Introduction

For a number field F and an odd prime p, many authors investigated the p-rank of the tame kernel $K_2\mathcal{O}_F$. It has been shown that this value is related to the order of the ideal class group of some related number field. So, some relations of p-ranks or the p-Sylow

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subgroups between the tame kernel $K_2\mathcal{O}_F$ and the ideal class group of $F(\zeta_p)$ have been established, see Tate [13], Keune [7], Browkin [1], Qin [11] and others [3,4,8–10,12], etc.

In particular, using an appropriate reflection theorem, Browkin [1] proved several results connecting the p-rank$(K_2\mathcal{O}_F)$ with the p-rank$(\mathrm{Cl}(\mathcal{O}_E))$, where E is an appropriate subfield of $F(\zeta_p)$, under the assumption that the fields F and $\mathbb{Q}(\zeta_p)$ are linearly disjoint, i.e. $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. In the case of quadratic fields F and $p = 3, 5$, the results are more explicit. In [11], Qin set up a new reflection theorem which generalizes the classical Scholz theorem (see [14]), and then he presented some formulas for p-rank$(K_2\mathcal{O}_F)$, which also connect the p-rank$(K_2\mathcal{O}_F)$ with the p-rank$(\mathrm{Cl}(\mathcal{O}_E))$ for some appropriate subfield E of $F(\zeta_p)$ under similar assumption. Qin also considered the p-Sylow subgroup of $K_2\mathcal{O}_F$ with F being some special extension of a quadratic field. On the other hand, Zhou [15] once considered the p-rank$(K_2\mathcal{O}_F)$ when F is a multi-quadratic field, and Browkin [2] studied the case of cyclic cubic fields.

In the present paper, we consider the general number field case and as results, nearly all of Browkin's results about quadratic fields in [1] are generalized to those for general number fields. As an example, we consider the case of cyclic quartic fields in more details. More precisely, using the results on cyclic quartic fields [5,6] and also using Qin's result we give some results connecting the p-rank$(K_2\mathcal{O}_F)$ with the p-rank$(\mathrm{Cl}(\mathcal{O}_E))$, where F is a cyclic quartic field and E is an appropriate subfield of $F(\zeta_p)$.

In Section 2, we consider the case of general number fields and Browkin's results are improved, in particular, a p-rank formula between the tame kernel and the ideal class group for totally real number field of odd degree is obtained when p is a Fermat prime. In Section 3, we prove some properties about cyclic quartic fields which are needed in the sequel sections. In Section 4, the computation of the values $|S'|$ for cyclic quartic fields is given. In Section 5, we give the p-ranks of tame kernels for cyclic quartic fields.

2. General number fields

Let p be an odd prime number, and ζ_p a p-th primitive root of unity. Then

$$\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = \{\sigma_a : 1 \le a \le p-1\},$$

where $\sigma_a(\zeta_p) = \zeta_p^a$.

For a fixed primitive root $g \pmod{p}$, let $\sigma := \sigma_g$. Then $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is a cyclic group of order $p-1$ generated by σ.

In this section, we always assume that $F/\mathbb{Q}$ is a Galois extension of degree n such that

$$\zeta_p \notin F \quad \text{and} \quad F \supsetneq F \cap \mathbb{Q}(\zeta_p).$$

Let $K := F \cap \mathbb{Q}(\zeta_p)$. Then $l := [K : \mathbb{Q}] < n$.

Let ω be the Teichmüller character of the group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p)^*$. We use the symbol $\bar{\omega}$ to denote the restriction of ω on $G_1 := \mathrm{Gal}(\mathbb{Q}(\zeta_p)/K) = \langle \sigma^l \rangle$, i.e.

$$\bar{\omega} := \omega|_{G_1}.$$