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Resultants of minimal polynomials of maximal real cyclotomic extensions

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ABSTRACT

Define the nth real cyclotomic polynomial to be the minimal polynomial over $\mathbb{Z}$ of $\zeta_n + \zeta_n^{-1}$, where $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity. We prove that the real cyclotomic polynomials can be formed from compositions of polynomials closely related to the Chebyshev polynomials of the first kind. We use these relations to determine the resultant of two real cyclotomic polynomials.

1. Introduction

For each positive integer n, the nth cyclotomic polynomial Φ_n is the minimal polynomial over the integers of $\zeta_n = e^{2\pi i/n}$, and has roots equal to the primitive nth roots of unity. The cyclotomic fields $\mathbb{Q}(\zeta_n)$ are central objects of algebraic number theory, and have been extensively studied. Within each cyclotomic field, the subfield $\mathbb{Q}(\zeta_n) \cap \mathbb{R} = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is the maximal real subfield of $\mathbb{Q}(\zeta_n)$. We refer to the minimal

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polynomial over $\mathbb{Z}$ of $\zeta_n+\zeta_n^{-1}$ as the nth *real cyclotomic polynomial*, and denote it by Ψ_n. The resultant of two such polynomials is the focus of the present article.

This paper was inspired by the work of Apostol [1], who determined a concise formula for the resultant of two cyclotomic polynomials. In general, the resultant of two polynomials is computed using formulas involving the coefficients of the polynomials. Such calculations are often torturous to perform by hand, but thankfully they can be done quickly and easily by using computer software. Nevertheless, if one is working with polynomials from a prescribed family, it is desirable to have simple closed forms for their resultants.

Along these lines, Apostol showed that for $n > m > 1$, the resultant of Φ_m with Φ_n equals $p^{\phi(m)}$ if n/m is a power of a prime p, and 1 otherwise. More recently [2,4], formulas have been given for resultants involving the Chebyshev polynomials of the first and second kind. In this paper, we determine the resultants for the real cyclotomic polynomials Ψ_n.

The article is organized as follows. Section 2 discusses how the Ψ_n can be expressed in terms of compositions involving a family of polynomials connected to the Chebyshev polynomials of the first kind. In Section 3, we use Apostol's formula from [1] to easily calculate the absolute value of the resultant of Ψ_m with Ψ_n (Theorem 3.4), although determining its sign requires additional work. While we cannot obtain a formula as elegant as Apostol's, in Sections 4 and 5 we show that the sign can be determined using the Legendre symbol $\left(\frac{a}{p}\right)$ (Theorems 4.2, 4.3, 4.4, and Corollaries 5.6, 5.8). The theorems are proved by exploiting connections among the Ψ_n, the Φ_n, and the Chebyshev polynomials.

2. Composition rules for Ψ_n

We maintain the notation given in the introduction: for each $n \geq 1$, $\zeta_n = e^{2\pi i/n}$ is a primitive nth root of unity, and Φ_n is the nth cyclotomic polynomial, which is the minimal polynomial of ζ_n.

Definition 2.1. For each $n \geq 1$, define $\Psi_n \in \mathbb{Z}[x]$ to be the minimal polynomial of $2\cos(\frac{2\pi}{n}) = \zeta_n + \zeta_n^{-1}$. We call Ψ_n the nth *real cyclotomic polynomial*. For all $n \geq 1$ and all $d \in \mathbb{Z}$, define $\alpha_{n,d}$ to be the real number $\alpha_{n,d} = 2\cos(\frac{2\pi d}{n}) = \zeta_n^d + \zeta_n^{-d}$.

The first ten real cyclotomic polynomials are:

$$\Psi_1(x) = x-2 \qquad \Psi_2(x) = x+2$$

$$\Psi_3(x) = x+1 \qquad \Psi_4(x) = x$$

$$\Psi_5(x) = x^2+x-1 \qquad \Psi_6(x) = x-1$$

$$\Psi_7(x) = x^3+x^2-2x-1 \qquad \Psi_8(x) = x^2-2$$

$$\Psi_9(x) = x^3-3x+1 \qquad \Psi_{10}(x) = x^2-x-1$$