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Polynomial identities on eigenforms

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In this paper, we fix a polynomial with complex coefficients and determine the eigenforms for $SL_2(\mathbb{Z})$ which can be expressed as the fixed polynomial evaluated at other eigenforms. In particular, we show that when one excludes trivial cases, only finitely many such identities hold for a fixed polynomial.

1. Introduction

Identities between Hecke eigenforms often give rise to surprising relationships between arithmetic functions. A well-known example involves the sum of divisor functions $\sigma_3(n)$ and $\sigma_7(n)$:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j). \tag{1}$$

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This identity is easily derived from the fact that $E_4E_4 = E_8$, where E_4 (respectively E_8) is the weight 4 (respectively 8) Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. Any product relation among eigenforms gives rise to a similar identity. Duke [3] has shown that an eigenform for $\mathrm{SL}_2(\mathbb{Z})$ may be decomposed as a product of two others in only sixteen cases (independently observed by Ghate [5]). Ghate [6] later considered eigenforms of higher level, and showed that there are still only finitely many such decompositions as long as the level is squarefree, and the weights of all eigenforms are at least 3. Johnson [7] has recently extended this result by showing that only a finite number of decompositions involving eigenforms of weight at least 2 exist for any given level and character.

Emmons and Lanphier [4] considered product decompositions involving any number of eigenforms for $\mathrm{SL}_2(\mathbb{Z})$, and showed that the only relations that arise are the 16 identified by Duke and Ghate, and a few trivially implied by them. In fact, these results show that the product decompositions they describe occur only when dimension considerations require it.

In this paper we move from monomial to polynomial decompositions. Our main result is the following theorem:

Theorem 1.1. *For a fixed $P \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ there exist only finitely many $n+1$ tuples $(f_1, f_2, \ldots, f_n, h)$ of non-zero eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ such that*

$$P(f_1, f_2, \ldots, f_n) = h, \tag{2}$$

where the weights of all the f_i are less than the weight of h.

Note that in order to discuss polynomial relationships amongst eigenforms, the addition of these eigenforms must be defined. Thus, if we have a polynomial relation on eigenforms, each term of the polynomial will have the same weight.

Our proof considers a particular decomposition of an arbitrary eigenform, and obtains an upper bound for the weight of that form. For the Eisenstein series, we rely on the fact that we have an explicit formula for their Fourier coefficients. The cuspidal case is more difficult. Our proof relies on several number theoretic lemmas, the Hecke relation satisfied by the Fourier coefficients of eigenforms, and bounds on the magnitude of the Fourier coefficients of cuspidal eigenforms.

As mentioned above, it was observed by Duke [3], Ghate [6], and Emmons and Lanphier [4] that the only product decompositions of eigenforms over $\mathrm{SL}_2(\mathbb{Z})$ are those forced by dimension considerations. Extensive computations suggest that this is true for most, if not all, polynomial decompositions as well.

2. Notation and conventions

Throughout this paper, if f is a modular form over $\mathrm{SL}_2(\mathbb{Z})$, we will let $a_f(n)$ denote the nth Fourier coefficient of f. In other words, f has a Fourier expansion given by: