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Monogenity of totally real algebraic extension fields over a cyclotomic field

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ABSTRACT

Let K be a composite field of a cyclotomic field k_n of odd conductor $n \geqq 3$ or even one $\geqq 8$ with $4|n$ and a totally real algebraic extension field F over the rationals $\boldsymbol{Q}$ and both fields k_n and F are linearly disjoint over $\boldsymbol{Q}$ to each other. Then the purpose of this paper is to prove that such a relatively totally real extension field K over a cyclotomic field k_n has no power integral basis. Each of the composite fields K is also a CM field over the maximal real subfield $k_n^+ \cdot F$ of K. This result involves the previous work for $K = k_n \cdot F$ of the Eisenstein field $k_n = k_3$ and the maximal real subfields $F = k_{p^n}^+$ of prime power conductor p^n with $p \geqq 5$, and an analogue $K = k_n \cdot F$ of cyclotomic fields $k_n = k_{2^m}$ $(m \geqq 3)$ with a totally real algebraic fields F of $K = k_4 \cdot F$ with a cyclic cubic field F except for $k_4 \cdot k_7^+$ and $k_4 \cdot k_{3^2}^+$ of conductors 28 and 36.

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1. Introduction

Let L be an algebraic number field over the field $\boldsymbol{Q}$ of rational numbers with degree $[L : \boldsymbol{Q}] = n$. Z_L denotes the ring of integers in L. The ring Z_L has an integral basis $\{\omega_j\}_{0 \leqq j \leqq n-1}$ as a $\boldsymbol{Z}$-module $\boldsymbol{Z}[\omega_0, \cdots, \omega_{n-1}]$ of rank n, where $\boldsymbol{Z}$ denotes the ring of rational integers. If there exists an integer ξ such that $Z_L = \boldsymbol{Z}[1, \xi, \cdots, \xi^{n-1}]$, then we say that Z_L has a power integral basis or L is monogenic. To characterize a family $\mathcal{F}$ of algebraic number fields whose rings of integers have a power integral basis is known as a problem of Hasse [9,4,7,8]. If the family $\mathcal{F}_+$ consists of all the real quadratic fields or the maximal real subfields k_n^+ of cyclotomic fields k_n of conductor n, the fields in $\mathcal{F}_+$ are totally real and monogenic [6,13]. It is known that if the conductor n of k_n is odd, then for any subfield L of k_n, the ring Z_L of integers in L has a normal basis generated by the Gauß period η of length $\varphi(n)/[k_n : L]$ for a primitive nth root ζ of unity [5]. Here $\varphi(\cdot)$ denotes the Euler function. D.S. Dummit and H. Kisilevsky stated that there are infinitely many cyclic cubic fields, accordingly totally real fields which are monogenic [1]. However in the works [12,7], a field L which belongs to the family $\mathcal{F}_2$ of linearly disjoint abelian extension fields $k \cdot F$ of imaginary quadratic fields k and certain real abelian fields F, Z_L does not have any power integral basis. Each field L in $\mathcal{F}_2$ is recognized as a relatively totally real abelian extension field over an imaginary quadratic field [Theorem 2, Theorem 3]. Consider the family $\mathcal{F}_\pm$ of cyclotomic fields k_n of odd conductor $n \geqq 3$ or even one $n \geqq 8$ with $4|n$ and totally real number fields F. Assume that their discriminants of k_n and F are coprime. The aim of this article is to extend our previous results by proving that the composite field $k_n \cdot F$ is not monogenic. Here each field in $\mathcal{F}_\pm$ makes a CM field over the totally real maximal subfield $k_n^+ \cdot F$ of L. The related works are found in [2,3,11].

2. Theorem and known results

We claim that

Theorem 1. *Let K be a composite field $k_n \cdot F$ of a cyclotomic field k_n of odd conductor $n \geqq 3$ or even one $n \geqq 8$ with $4|n$ and a totally real algebraic number field F distinct from the rationals $\boldsymbol{Q}$. Assume that their field discriminants are coprime. Then K is non-monogenic.*

For the following families of the fields K, the monogenity of the rings Z_K of integers in K has been characterized. Theorem 1 involves Theorem 2 and Theorem 3(2) as special families.

Theorem 2. *(See [12].) Let K be the composite field of the Eisenstein field k_3 and the maximal real subfield $k_{p^n}^+$ of prime power conductor p^n with $p \geqq 5$. Then the ring Z_K of integers in K does not have a power integral basis.*