Journal of Number Theory 158 (2016) 365–396

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Expansions of generalized Euler's constants into the series of polynomials in π^{-2} and into the formal enveloping series with rational coefficients only

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ARTICLE INFO

Article history:
Received 1 January 2015
Received in revised form 26 June 2015
Accepted 29 June 2015
Available online 18 August 2015
Communicated by David Goss

Keywords:
Generalized Euler's constants
Stieltjes constants
Stirling numbers
Factorial coefficients
Series expansion
Divergent series
Semi-convergent series
Formal series
Enveloping series
Asymptotic expansions
Approximations
Bernoulli numbers
Harmonic numbers
Rational coefficients
Inverse pi

ABSTRACT

In this work, two new series expansions for generalized Euler's constants (Stieltjes constants) γ_m are obtained. The first expansion involves Stirling numbers of the first kind, contains polynomials in π^{-2} with rational coefficients and converges slightly better than Euler's series $\sum n^{-2}$. The second expansion is a semi-convergent series with rational coefficients only. This expansion is particularly simple and involves Bernoulli numbers with a non-linear combination of generalized harmonic numbers. It also permits to derive an interesting estimation for generalized Euler's constants, which is more accurate than several well-known estimations. Finally, in Appendix A, the reader will also find two simple integral definitions for the Stirling numbers of the first kind, as well an upper bound for them.

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http://dx.doi.org/10.1016/j.jnt.2015.06.012

1. Introduction and notations

1.1. Introduction

The ζ-function, which is usually introduced via one of the following series,

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}\,, & \operatorname{Re} s > 1 \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\,, & \operatorname{Re} s > 0\,, \quad s \neq 1 \end{cases} \tag{1}$$

is of fundamental and long-standing importance in modern analysis, number theory, theory of special functions and in a variety other fields. It is well known that $\zeta(s)$ is meromorphic on the entire complex s-plane and that it has one simple pole at $s = 1$ with residue 1. Its expansion in the Laurent series in a neighbourhood of $s = 1$ is usually written the following form

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \frac{(-1)^m (s-1)^m}{m!} \gamma_m\,, \qquad s \neq 1, \tag{2}$$

where coefficients γ_m, appearing in the regular part of expansion (2), are called *generalized Euler's constants* or *Stieltjes constants*, both names being in use.[2,3] Series (2) is the standard definition for γ_m. Alternatively, these constants may be also defined via the following limit

$$\gamma_m = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{\ln^m k}{k} - \frac{\ln^{m+1} n}{m+1} \right\}, \quad m = 0, 1, 2, \ldots \tag{3}$$

The equivalence between definitions (2) and (3) was demonstrated by various authors, including Adolf Pilz [69], Thomas Stieltjes, Charles Hermite [1, vol. I, letter 71 and following], Johan Jensen [87,89], Jérôme Franel [56], Jørgen P. Gram [69], Godfrey H. Hardy [73], Srinivasa Ramanujan [2], William E. Briggs, S. Chowla [24] and many others, see e.g. [16,176,84,128]. It is well known that $\gamma_0 = \gamma$ Euler's constant, see e.g. [128],

[2] The definition of Stieltjes constants accordingly to formula (2) is due to Godfrey H. Hardy. Definitions, introduced by Thomas Stieltjes and Charles Hermite between 1882–1884, did not contain coefficients $(-1)^m$ and $m!$ In fact, use of these factors is not well justified; notwithstanding, Hardy's form (2) is largely accepted and is more frequently encountered in modern literature. For more details, see [1, vol. I, letter 71 and following], [110, p. 562], [19, pp. 538–539].

[3] Some authors use the name *generalized Euler's constants* for other constants, which were conceptually introduced and studied by Briggs in 1961 [23] and Lehmer in 1975 [114]. They were subsequently rediscovered in various (usually slightly different) forms by several authors, see e.g. [173,140,190]. Further generalization of both, generalized Euler's constants defined accordingly to (2) and generalized Euler's constants introduced by Briggs and Lehmer, was done by Dilcher in [49].