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Rational products of singular moduli

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ABSTRACT

We show that with "obvious" exceptions the product of two singular moduli cannot be a non-zero rational number. This gives a totally explicit version of André's 1998 theorem on special points for the hyperbolas $x_1x_2 = A$, where $A \in \mathbb{Q}$.

1. The result

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ be the Poincaré plane and j the j-invariant. The numbers of the form $j(\tau)$, where $\tau \in \mathbb{H}$ is an imaginary quadratic number, are called *singular moduli*. It is known that $j(\tau)$ is an algebraic integer satisfying

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$$[\mathbb{Q}(\tau, j(\tau)) : \mathbb{Q}(\tau)] = [\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(\Delta),$$

where Δ is the discriminant of the complex multiplication order $\mathcal{O} = \operatorname{End}\langle\tau, 1\rangle$ (the endomorphism ring of the lattice generated by τ and 1) and $h(\Delta) = h(\mathcal{O})$ is the class number.

Let $F(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ be an irreducible complex polynomial with

$$\deg_{x_1} F + \deg_{x_2} F \geq 2.$$

In 1998 André [2] proved that the equation $F(j(\tau_1), j(\tau_2)) = 0$ has at most finitely many solutions in singular moduli $j(\tau_1), j(\tau_2)$, unless $F(x_1, x_2)$ is the classical modular polynomial $\Phi_N(x_1, x_2)$ of some level N. Recall that Φ_N is defined (up to a constant multiple) as the irreducible polynomial satisfying $\Phi_N(j, j_N) = 0$, where $j_N(z) = j(Nz)$.

André's result was the first non-trivial contribution to the celebrated André–Oort conjecture on the special subvarieties of Shimura varieties; see [7,11] and the references therein.

Independently of André the same result was also obtained by Edixhoven [6], but Edixhoven had to assume the Generalized Riemann Hypothesis for certain L-series to be true. See also the work of Breuer [4], who gave an explicit version of Edixhoven's result.

Further proofs followed; we mention specially the remarkable argument of Pila [10]. It is based on an idea of Pila and Zannier [12] and readily extends to higher dimensions [11].

The arguments of André and Pila are non-effective, because they use the Siegel–Brauer lower bound for the class number. Recently Kühne [8,9] and, independently, Bilu, Masser, and Zannier [3] found unconditional effective proofs of André's theorem. Besides giving general results, both articles [9] and [3] treat also some particular curves, showing they have no CM-points at all. For instance, Kühne [9, Theorem 5] proved that a sum of two singular moduli can never be 1:

$$j(\tau_1) + j(\tau_2) \neq 1. \tag{1}$$

Neither can their product be 1, as shown in [3]:

$$j(\tau_1)j(\tau_2) \neq 1. \tag{2}$$

A vast generalization of (1) was given in [1]: it is shown that, with "obvious" exceptions, two distinct singular moduli cannot satisfy a linear relation over $\mathbb{Q}$.

In this note we combine ideas from [1] and [3] and generalize (2), showing that, with "obvious" exceptions, the product of two singular moduli cannot be a non-zero rational number.

Theorem 1.1. *Assume that $j(\tau_1)j(\tau_2) \in \mathbb{Q}^\times$. Then we have one of the following options:*

- *(rational case) both $j(\tau_1)$ and $j(\tau_2)$ are rational numbers (in fact integers);*
- *(quadratic case) $j(\tau_1)$ and $j(\tau_2)$ are of degree 2 and conjugate over $\mathbb{Q}$.*