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Generalized Markoff equations and Chebyshev polynomials

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A B S T R A C T

The Markoff equation is $x^2 + y^2 + z^2 = 3xyz$, and all of the positive integer solutions of this equation occur on one tree generated from $(1, 1, 1)$, called the Markoff tree. In this paper, we consider trees of solutions to $x^2+y^2+z^2 = xyz+A$. We say a tree satisfies the unicity condition if the maximum element of an ordered triple in the tree uniquely determines the other two. The unicity conjecture says that the Markoff tree satisfies the unicity condition. In this paper, we show that there exists a sequence of real numbers $\{c_n\}$ such that each tree generated from $(1, c_n, c_n)$ satisfies the unicity condition, and that these trees converge to the Markoff tree. We accomplish this by recasting polynomial solutions as linear combinations of Chebyshev polynomials, showing that these polynomials are distinct, and evaluating them at certain values.

0. Introduction

In 1879 and 1880 [Mar79,Mar80], A. Markoff used continued fractions to show that there is a one-to-one correspondence between the indefinite quadratic forms with minima

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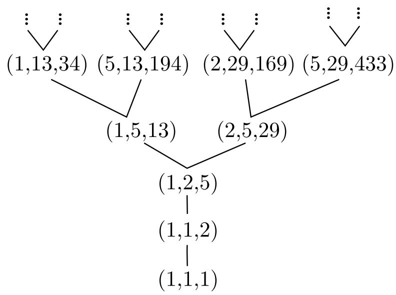

Fig. 1. The Markoff tree $\mathfrak{M}$.

greater than $\frac{1}{3}$ of the square root of its discriminant, and integral solutions to the following equation:

$$x^2 + y^2 + z^2 = 3xyz.$$

This equation is known as the Markoff equation. Many surprising connections have been made between integer solutions of the Markoff equation and various fields of mathematics, such as algebraic number theory, combinatorics, diophantine approximation, and hyperbolic geometry ([Aig13] is an excellent source for details of these connections). There is also a connection between integer solutions of the Markoff equation and exceptional representative sheaves on the complex projective plane $\mathbb{P}^2$ [Rud89]. We call $(x, y, z) \in \mathbb{R}^3$ an *ordered triple* if $x \leq y \leq z$, and we call (x, y, z) a *Markoff triple* if it is an ordered triple solution to the Markoff equation with x, y, and z all positive integers. If (x, y, z) is a Markoff triple then $(x, z, 3xz - y)$ and $(y, z, 3yz - x)$ are Markoff triples as well. Thus, any ordered triple solution creates additional solutions, forming a tree of solutions for the Markoff equation. In particular, we can generate a tree of solutions from $(1, 1, 1)$, which we refer to as the Markoff tree $\mathfrak{M}$, as shown in Fig. 1. Markoff used a method of descent to show that every Markoff triple descends all the way down to $(1, 1, 1)$ in a finite number of steps. Thus, all of the Markoff triples appear in $\mathfrak{M}$.

In 1913, Frobenius conjectured that the largest entry z of a Markoff triple uniquely determines the other two [Fro13]. It is common to say that the maximum element is *unique* when this is the case. This is known as the unicity conjecture and remains unsolved. Some partial results of the unicity conjecture have been settled. It is known that if z or $3z \pm 2$ is a prime, twice a prime, or four times a prime then it is unique [Bar96]; or if z is a prime power then it is unique [But01]. Currently, it is known that z is unique if $z = k \cdot p^\beta$, with $k \leq 10^{35}$, p prime and k relatively prime to p [But01]; and if $3z \pm 2 = k \cdot p^\beta$, with $k \leq 10^{10}$, p prime, and k relatively prime to p [CC13]. The upper bounds for k in these last two results are based on the empirical result that z is unique if $z < 10^{140}$ [Bar96].

There are several generalizations of the Markoff equation, including

$$x^2 + y^2 + z^2 = axyz + b,$$