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# On the largest prime factor of the ratio of two generalized Fibonacci numbers

Carlos Alexis Gómez Ruiz [a,*], Florian Luca [b]

[a] Departamento de Matemáticas, Universidad del Valle, 25360 Cali, Calle 13 No 100-00, Colombia
[b] School of Mathematics, University of the Witwatersrand, P.O. Box Wits 2050, South Africa

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A B S T R A C T

A generalization of the well-known Fibonacci sequence is the $k$-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ for some integer $k \geq 2$, whose first $k$ terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding $k$ terms. In this paper, we look at the prime factors of the reduced rational number $F_n^{(k)}/F_m^{(\ell)}$ as $\max\{m, n, k, \ell\}$ tends to infinity.



## 1. Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ satisfies the recurrence $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$, with the initial values $F_0 = 0$, $F_1 = 1$. A well-known generalization of this is

* Corresponding author.
*E-mail addresses:* carlos.a.gomez@correounivalle.edu.co (C.A. Gómez Ruiz), Florian.Luca@wits.ac.za (F. Luca).

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the $k$-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$, which satisfies the $k$-th order linear recurrence

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)} \qquad (n \geq 2-k),$$

with the $k$ initial values $0, 0, \ldots, 1$. Observe that

$$F_1^{(k)} = 1,\ F_2^{(k)} = 1,\ F_3^{(k)} = 2, \ldots,\ F_{k+1}^{(k)} = 2^{k-1}, \quad \text{and} \quad F_{k+1}^{(k)} = 2^k - 1.$$

In particular, $F_j^{(k)}$ is a power of 2 for all $1 \leq j \leq k+1$. Such powers of 2 are called *trivial*. In [2], it was shown that there is no nontrivial power of 2 in the $k$-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq 1}$ for any $k \geq 3$. When $k = 2$, the only nontrivial power of 2 is $F_6^{(2)} = 8$. An extension of this result appears in [6], where it is proved that if $k \geq 2$ and $m \geq 1$, $n \geq 1$, $s \geq 0$ are such that

$$F_n^{(k)} = 2^s F_m^{(k)},$$

then either $s = 0$ (and $m = n$) or $F_m^{(k)}$ is a power of 2. In particular, if on the set $\{F_n^{(k)} : n \geq 1\}$ we put an equivalence relation under which $a$ is equivalent to $b$ if the ratio $a/b$ is a power of 2 of integer exponent, then there is an equivalence class formed by all powers of 2 (classified in [2]) and all other equivalence classes are singletons. In particular, if the largest odd factor of $F_n^{(k)}$ exceeds 1, then this largest odd factor uniquely determines $n$.

In [3], the equation

$$F_n^{(k)} = F_m^{(\ell)} \tag{1}$$

was studied. To avoid trivialities, it was assumed that $n \geq k+2$, $m \geq \ell + 2$ and $(n, k) \neq (m, \ell)$. Under these assumptions together with $k > \ell$, it was shown that the only solutions are

$$(n, k, m, \ell) = (6, 3, 7, 2),\ (11, 7, 12, 3).$$

For an integer $m$ put $P(m)$ for the maximal prime factor of $m$ with the convention that $P(0) = P(\pm 1) = 1$. For a rational number $a/b$ in reduced form, put $P(a/b)$ for $P(ab) = \max\{P(a), P(b)\}$. In [1], it was shown that the inequality

$$P(F_n^{(k)}) \geq 0.01\sqrt{\log n \log\log n} \tag{2}$$

holds for all $k \geq 2$ and $n \geq k+2$.

In this paper, we study a more general problem which in particular encompasses all of the above. Namely, for positive integers $n$, $m$, $k \geq 2$, $\ell \geq 2$ we write