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Conjectures on the evaluation of certain functions with algebraic properties

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ABSTRACT

In this article we consider functions with Moebius-periodic Taylor expansion coefficients. These functions under some conditions take algebraic values and can be evaluated in terms of theta functions and the Dedekind eta function. Special cases are the elliptic singular moduli, the Rogers–Ramanujan continued fraction, Eisenstein series and functions associated with Jacobi symbol coefficients.

1. Introduction

A number of functions defined over the complex plane, which are important in Number Theory and other disciplines, have been found to have real algebraic values at certain points. The aim of this note is to explore properties common to these functions which appear to be connected to this important feature. We begin by reviewing some relevant material.

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An elliptic singular modulus k_r is the solution x of the equation

$$\frac{K(\sqrt{1-x^2})}{K(x)} = \sqrt{r} \tag{1}$$

where

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} x^{2n} = \frac{2}{\pi} K(x) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2(\phi)}} \tag{2}$$

The 5th degree modular equation, which connects k_{25r} and k_r, is (see [16], Chapter 19, pp. 280–288):

$$k_r k_{25r} + k'_r k'_{25r} + 2^{5/3}(k_r k_{25r} k'_r k'_{25r})^{1/3} = 1 \tag{3}$$

The problem of solving (3) and finding k_{25r} reduces to that of solving, what Hermite termed the depressed equation (see [3], Section 10.11, pp. 314–316):

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0 \tag{4}$$

where $u = k_r^{1/4}$ and $v = k_{25r}^{1/4}$.

The function k_r is also connected to null theta functions from the relations

$$k_r = \frac{\theta_2^2(q)}{\theta_3^2(q)}, \text{ where } \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \text{ and } \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{5}$$

$q = e^{-\pi\sqrt{r}}$.

In these terms a closed form solution of the depressed equation is

$$k_{25r} = \frac{\theta_2^2(q^5)}{\theta_3^2(q^5)} \tag{6}$$

But this is not satisfactory.

For example, in the case of Ramanujan's famous π formulas (see [13] and related references), one has to obtain from the exact value of k_r the exact value of k_{25r} in radicals. (Here we mention the fact that when r is positive rational then the value of k_r is an algebraic number.) Another example is the Rogers–Ramanujan continued fraction (RRCF)

$$R(q) = \frac{q^{1/5}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \cdots \tag{7}$$

(see [4,5,7–9,11,18,21] and [16], Chapter 16, pp. 77–86, [17], Chapter 32, pp. 12–45), the value of which also depends on the depressed equation.