Journal of Number Theory 155 (2015) 100–110

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Self-similar aperiodic patterns with 9-fold symmetry

H.J. Rivertz

Sør-Trøndelag University College, 7004 Trondheim, Norway

ARTICLE INFO

Article history:
Received 22 September 2014
Received in revised form 10 March 2015
Accepted 17 March 2015
Available online 6 May 2015
Communicated by David Goss

MSC:
52C20
52C23
11R18

Keywords:
Penrose-type tilings
Cyclotomic fields
Galois group
Units

ABSTRACT

We show there exists one planar self-similar tiling with D_9 symmetry. The tiles are the regular hexagon and the rhombi with 20, 40, or 80 degree angles. It has two almost isometric tilings with D_3 symmetry.

1. Introduction

Periodic tilings in the plane can only have symmetries of order 2, 3, 4, or 6. In the early 1980's, Shechtman discovered solids that showed diffraction patterns with 5 and 10 fold symmetry [5]. These solids, the so-called quasi-crystals, have a non-periodic but strictly ordered structure. Aperiodic tilings model such structures.

E-mail address: hans.j.rivertz@hist.no.

http://dx.doi.org/10.1016/j.jnt.2015.03.010

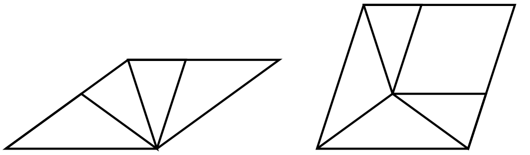

Fig. 1. Subdivision of the tiles.

In the 1970's, Roger Penrose discovered a tiling of the entire plane with 5 fold symmetry [4]. It consists of two different rhombic tiles, one thin with a 36 degree angle, and one fat with a 72 degree angle as shown in Fig. 1. This tiling has remarkable properties, it has a smaller subset of the vertices that is an exact, enlarged copy of the original set of vertices, and the property of inflation, where each tile can be subdivided into smaller tiles of the same type.

The Penrose tilings come from a cut and project method [1]. I.e., a subset of the integer grid $\mathbb{Z}^5$ is projected onto a 2-dimensional affine subspace E in $\mathbb{R}^5$. This subset consists of those integer grid points lying in the strip $\Sigma_E = E + (0,1)^5$. The members of this subset are called admissible points.

The vector space $\mathbb{R}^5$ can be written as a sum of three linear subspaces E_1, E_2, and D that are invariant under cyclic permutations of the coordinates. E_1 and E_2 have dimension two while D has dimension one. D is spanned by the vector $\mathbf{d} = (1,1,1,1,1)$ and is a rational subspace. Both E_1 and E_2 are irrational subspaces but their sum $W = E_1 + E_2$ is a rational subspace.

The affine subspace E is a parallel translation of E_1 in W, that is, $E = E_1 + \mathbf{a}$ where $\mathbf{a} \in W$. We assume that E is in general position, which means that there are no points in E with more than two integer coordinates.

When we replace $\mathbb{R}^5$ with $\mathbb{R}^n$ for suitable n, we get similar aperiodic patterns. The case when n is any prime larger than or equal to 5 is treated in [3]. The special case where $n = 7$ is discussed in [2]. In this article we look at the case where $n = 9$. The tiles are the regular hexagon and the rhombi with 20, 40, or 80 degree angles.

2. Tiling with D_n symmetry

First, we consider the case where $n = 2r + 1$. The dihedral group D_n acts on $\mathbb{R}^n$ by permutations of the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. $\mathbb{R}^n$ is decomposed by r 2-dimensional subrepresentations $E_1, E_2, \ldots, E_r$ and one 1-dimensional fixed space $D = \mathbb{R}\mathbf{d}$, where $\mathbf{d} = (1, \ldots, 1) \in \mathbb{R}^n$. The 2-dimensional subrepresentations are given by $E_j = \{z\mathbf{w}_j + \bar{z}\bar{\mathbf{w}}_j | z \in \mathbb{C}\}$, $j = 1, 2, \ldots, r$, where $\mathbf{w}_j = (\zeta^j, \zeta^{2j}, \cdots, \zeta^{nj}) \in \mathbb{R}^n$ and $\zeta = e^{2\pi i/n}$. We define $W = \sum_{\gcd(i,n)=1}^{1\le i\le r} E_i$ and its orthogonal complement $W^\perp = \sum_{p|n}^{1<p<n} F_p$, where $F_p = D + \sum_{j=1}^{\lfloor r/p \rfloor} E_{pj}$, $p|n$, $1 < p < n$. A basis for F_p is given by $\mathbf{f}_{i,p} = \sum_{j=0}^{p-1} \mathbf{e}_{i+nj/p}$, $i = 1, 2, \ldots, n/p$.

Lemma 1. *W is an irreducible rational D_n-submodule of $\mathbb{R}^n$.*