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Distribution of cusp sections in the Hilbert modular orbifold ☆,☆☆

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ABSTRACT

Text. Let K be a number field, let $\mathcal{M}$ be the Hilbert modular orbifold of K, and let m_q be the probability measure uniformly supported on the cusp cross sections of $\mathcal{M}$ at height q. We show that m_q distributes uniformly with respect to the normalized Haar measure m on $\mathcal{M}$ as q tends to zero, and relate the rate by which m_q approaches m to the Riemann hypothesis for the Dedekind zeta function of K.

Video. For a video summary of this paper, please visit http://youtu.be/_39k9paBQjM.

1. Introduction

Let $\mathscr{H}_2 = \{ x + iy \in \mathbb{C} \mid y > 0 \}$ be the Poincaré upper half-plane with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. The group $PSL(2, \mathbb{R})$ acts on $\mathscr{H}_2$ by fractional linear transformations which, as well, are hyperbolic isometries. The modular group $PSL(2, \mathbb{Z})$ is a discrete subgroup of $PSL(2, \mathbb{R})$ and the quotient space $PSL(2, \mathbb{Z})\backslash\mathscr{H}_2$ is the classical modular orbifold. From the classification of horocycles it follows that for each $y > 0$, the

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modular orbifold has a unique closed horocycle $\mathcal{C}_y$ of length y^{-1}. Let m_y be the probability measure uniformly supported on $\mathcal{C}_y$ (w.r.t. arc length). Let m be the normalized hyperbolic measure of $PSL(2,\mathbb{Z})\backslash\mathscr{H}_2$. We have the following well-known result due to D. Zagier (cf. [35]).

Theorem 1. *Let f be a smooth function on $PSL(2,\mathbb{Z})\backslash\mathscr{H}_2$ with compact support. Then,*

$$m_y(f) = m(f) + o(y^{1/2-\epsilon}) \qquad (y \to 0)$$

for all $0 < \epsilon < 1/2$.

In particular, the measures m_y converge vaguely to m as $y \to 0$. Besides, in [35] Zagier establishes the following remarkable equivalence to the Riemann hypothesis.

Theorem 2. *The Riemann hypothesis holds if and only if for every smooth function f with compact support on $PSL(2,\mathbb{Z})\backslash\mathscr{H}_2$ one has*

$$m_y(f) = m(f) + o(y^{3/4-\epsilon}) \qquad (y \to 0)$$

for all $0 < \epsilon < 3/4$.

The unit tangent bundle of $PSL(2,\mathbb{Z})\backslash\mathscr{H}_2$ can be identified with $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ and the horocycle flow on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ is given by right multiplication with $\left[\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right]$. The horocycle flow on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ has a unique closed orbit $\widetilde{\mathcal{C}}_y$ of length y^{-1} for each $y > 0$. In the related work [23], P. Sarnak proved an analogue of Theorems 1 and 2 for $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$, and he also showed that the analogue of Theorem 1 holds for general non-compact hyperbolic surfaces of finite area. Likewise, in [31] A. Verjovsky has shown that the analogous estimate of Theorem 1 in Sarnak's work is optimal for certain characteristic functions on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$ and he also found a discrete measure on $\mathbb{R}$ related to this problem (cf. [32]). Many other aspects of this more general problem of equidistribution of closed horospheres for general cofinite hyperbolic manifolds have been considered by many authors. As a guide to the reader we mention Hejhal [14], Flaminio and Forni [10], Eskin and McMullen [8]. We also quote Refs. [5,28,26]. Finally, Cacciatori and Cardella have generalized Zagier's work to the case of certain horospheres in Siegel modular orbifolds [2].

The purpose of this article is to apply Zagier's theory to the case of a number field. Let us briefly describe our results. Given a number field K of degree $n = r_1 + 2r_2$ with ring of integers $\mathfrak{o}$, there exists a Riemannian manifold $\mathcal{H} = (\mathscr{H}_2)^{r_1} \times (\mathscr{H}_3)^{r_2}$ where the Hilbert modular group $\Gamma = PSL(2,\mathfrak{o})$ acts properly and discontinuously. If the field K has class number h, then the Hilbert modular orbifold $\mathscr{M} = \Gamma\backslash\mathcal{H}$ has h cusps and each cusp can be parametrized by the standard cusp at infinity. From the geometry of $\mathscr{M}$, it follows that, for each cusp λ and $q > 0$, there exist a *generalized closed horosphere* $\Gamma_\lambda\backslash B(q,\lambda)$ of dimension $2r_1 + 3r_2 - 1$ and volume $q^{-1}c$, where c is a certain constant