Journal of Number Theory 150 (2015) 21–25

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A Salem number with degree 34 and trace −3

S. El Otmani, G. Rhin, J.-M. Sac-Épée *

IECL, Université de Lorraine, Metz, France

ARTICLE INFO

Article history:
Received 23 June 2014
Received in revised form 17 October 2014
Accepted 3 November 2014
Available online 7 January 2015
Communicated by David Goss

MSC:
90C11
11C08
11R09
11S05
12D10
26C10

Keywords:
Salem numbers of small trace
Totally positive algebraic integers

ABSTRACT

In this paper, we give a new Salem number with degree 34 and trace −3, whereas the Salem number of lower degree known up to now, with trace −3, was of degree 54. The method used to find this Salem number also provides a list of 41 monic integer polynomials of smallest possible trace, irreducible, with degree 17, and having all roots real and positive. Moreover, we think that this list may be complete.

1. Introduction

A Salem number is an algebraic integer $\tau > 1$ of degree $n \geq 4$, whose other conjugates, except for τ^{-1}, have modulus 1. Its minimal polynomial $P = x^n + a_1x^{n-1} + \cdots + a_n$ is reciprocal, i.e., $x^nP(1/x) = P(x)$. P is called *a Salem polynomial of degree* $n = 2d$. The

* Corresponding author.
E-mail addresses: elotmani5@univ-lorraine.fr (S. El Otmani), georges.rhin@univ-lorraine.fr (G. Rhin), jean-marc.sac-epee@univ-lorraine.fr (J.-M. Sac-Épée).

http://dx.doi.org/10.1016/j.jnt.2014.11.013

trace of τ is $-a_1$. The Lehmer's Salem number is the positive root $1.176280818\ldots$ of the polynomial: $L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. It is the smallest known Salem number and has trace -1.

A classical question was the following: **do there exist Salem numbers of negative trace below -1?**

J. McKee and C.J. Smyth proved in 2005 that there are Salem numbers of every trace [7] and proved the following theorem.

Theorem. *For every negative integer $-T$ there is a Salem number of trace $-T$ and degree at most exp exp$(22 + 4T \log T)$.*

Then J. McKee and C.J. Smyth in 2004 [6] found a Salem number of degree 20 and trace -2. They used the relation between Salem numbers and totally positive algebraic integers. In 2011 J. McKee gave [5] a Salem number of degree 54 and trace -3 using the interlacing method [7]. In this work, we give a Salem number of degree 34 and trace -3 and it is expected of smallest possible degree.

Relations between Salem numbers and totally positive algebraic integers. In a Salem polynomial P with degree $2d$, we make the change of variable $z = x + 1/x + 2$. Then we get a monic integer polynomial Q of degree d whose all roots are real and positive, $d - 1$ are in $(0, 4)$ and one, i.e., $\tau + \tau^{-1} + 2$ is > 4. Its trace is $2d - a_1$. Thus, here we are interested in finding totally positive integer polynomials with degree $d = 17$ and trace $2d - 3 = 31$. The first step is to localize the zeros of Q. We use the classical method of explicit auxiliary functions. This method was introduced into number theory by C.J. Smyth [10] to study the absolute trace of totally positive algebraic integer α of degree d which is trace$(\alpha)/d$. This is the Schur–Siegel–Smyth trace problem. For an account of this method before 2006 see [1]. We use a very effective version of auxiliary function due to V. Flammang [3] to prove that the polynomials Q (degree 17 and trace 31) necessarily have their 17 roots in the interval $(0, 6.69)$.

The problem of establishing lists of degree d monic irreducible polynomials with integer coefficients, only positive real roots and minimal trace is a classical question in Number Theory. In [2] we have obtained 11 degree 16 polynomials with trace $29 = 2 * 16 - 3$ (none was found with degree 15). The computer program developed at that time was written in C++, and used the callable GLPK library which supplies a solver for linear programming. This program worked well up to degree 16. Unfortunately, for higher degrees, the program often crashed, looping infinitely with obscure messages about numerical instability in the primal simplex phase. For us, there was no need to seek further afield at that time since our objective was above all to find degree 16 polynomials with trace 29 (none was found for degree 15 and trace 27). For the purpose of this article, it became essential to be able to handle higher degrees, because the goal was to find Salem numbers, and none was found with degree 16. Then, we chose to use the implementation of GLPK in the Python framework, which allows a very good control on the progress of the calculation, via control parameters passed to the dedicated functions.