Journal of Number Theory 150 (2015) 41–46

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Perfect power Riesel numbers

Carrie Finch [a], Lenny Jones [b,*]

[a] *Mathematics Department, Washington and Lee University, Lexington, VA 24450, United States*

[b] *Department of Mathematics, Shippensburg University, Shippensburg, PA 17257, United States*

A R T I C L E   I N F O

Article history:
Received 1 August 2014
Received in revised form 26 October 2014
Accepted 2 November 2014
Available online 7 January 2015
Communicated by David Goss

MSC:
11B83
11Y05

Keywords:
Riesel number
Covering

A B S T R A C T

A Riesel number k is an odd positive integer such that $k\cdot 2^n - 1$ is composite for all integers $n \geq 1$. In 2003, Chen proved that there are infinitely many Riesel numbers of the form k^r, when $r \not\equiv 0, 4, 6, 8 \pmod{12}$, and he conjectured that such Riesel powers exist for all positive integers r. In 2008, Filaseta, Finch and Kozek extended Chen's theorem slightly by constructing Riesel numbers of the form k^4 and k^6. In 2009, Wu and Sun provided more evidence to support Chen's conjecture by showing that there exist infinitely many Riesel numbers of the form k^r for any positive integer r that is coprime to 15 015. In this article, we extend the results of Wu and Sun by proving that there exist infinitely many Riesel numbers of the form k^r for any positive integer r that is coprime to 105.

1. Introduction

A Riesel number k is an odd positive integer with the property that $k \cdot 2^n - 1$ is composite for all natural numbers n. The smallest known Riesel number is 509 203; indeed, H. Riesel [9] showed that if $k \equiv 509\,203 \pmod{11\,184\,810}$, then k is a Riesel

* Corresponding author.
E-mail address: lkjone@ship.edu (L. Jones).

http://dx.doi.org/10.1016/j.jnt.2014.11.004

number. It is believed that 509 203 is the smallest Riesel number. As of this writing, there are 50 odd positive integers smaller that 509 203 that are still candidates. See www.prothsearch.net/rieselprob.html for the most up-to-date information.

In 2003, Y.G. Chen [2] showed that there are perfect power Riesel numbers for certain powers. In particular, for values of r that are either odd or twice an odd number not divisible by 3, he constructed integers k such that $k^r \cdot 2^n - 1$ is composite for all natural numbers n. Furthermore, Chen conjectured that there are Riesel numbers that are perfect r-th powers for any positive integer r. In 2008, Filaseta, Finch and Kozek [5] extended Chen's theorem slightly by proving for each $n \in \{4, 6\}$ that there exists a set $\mathcal{T}_n$ of positive density such that each element in $\mathcal{T}_n$ is a Riesel number of the form k^r with $r \equiv 0 \pmod{n}$. Wu and Sun [11] provided further evidence in 2009 to support Chen's conjecture by showing that there exist infinitely many Riesel numbers of the form k^r for any positive integer r that is coprime to 15 015. In this paper, we extend the result of Wu and Sun by establishing the following theorem.

Theorem 1.1. *For any positive integer r with $\gcd(r, 105) = 1$, there exist infinitely many odd positive integers k such that $k^r \cdot 2^n - 1$ is composite for all integers $n \geq 1$. Moreover, $k^r \cdot 2^n - 1$ has at least two distinct prime divisors for each value of n, when $r \geq 4$.*

2. Preliminaries

The following concept, due to Erdős [4], is crucial to the proof of Theorem 1.1.

Definition 2.1. A *covering* of the integers is a finite system of congruences $x \equiv a_i \pmod{m_i}$, where $m_i > 1$, such that every integer n satisfies at least one of the congruences. For brevity of notation, we present a covering $\mathcal{C}$ as a set of ordered pairs (a_i, m_i). We let $\mathcal{L}_{\mathcal{C}}$ denote the least common multiple of all the moduli m_i occurring in $\mathcal{C}$.

Quite often when a covering $\mathcal{C}$ is used to solve a problem, there is a set of prime numbers associated with $\mathcal{C}$. In the situation occurring in this article, for each $(a_i, m_i) \in \mathcal{C}$, there exists a corresponding prime p_i, such that $2^{m_i} \equiv 1 \pmod{p_i}$, where $2^s \not\equiv 1 \pmod{p_i}$ for all positive integers $s < m_i$. We call such a prime a *primitive divisor* of $2^{m_i} - 1$. In terms of group theory, a primitive divisor p of $2^m - 1$, where $m > 1$ is an integer, is a prime such that in the group of units modulo p, which we denote $(\mathbb{Z}_p)^*$, the element 2 has order m. We denote the order of the integer z modulo a prime p as $\operatorname{ord}_p(z)$.

A covering with certain restrictions on the moduli is used to establish Theorem 1.1. To build this covering, we can use a particular modulus $m > 1$ as many times as there are distinct primitive divisors of $2^m - 1$. It is well known that $2^m - 1$, with $m > 1$, has at least one primitive divisor as long as $m \neq 6$. This result is originally due to Bang [1].

Two additional facts are needed here. The first result is due to Darmon and Granville [3].