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On the Erdős–Turán conjecture ☆

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ABSTRACT

Text. Let $\mathbb{N}$ be the set of all nonnegative integers and $k \geq 2$ be a fixed integer. For a set $A \subseteq \mathbb{N}$, let $r_k(A,n)$ denote the number of solutions of $a_1 + \cdots + a_k = n$ with $a_1, \ldots, a_k \in A$. In this paper, we prove that for given positive integer u, there is a set $A \subseteq \mathbb{N}$ such that $r_k(A,n) \geq 1$ for all $n \geq 0$ and the set of n with $r_k(A,n) = k!u$ has density one. This generalizes recent results of Chen and Yang.

Video. For a video summary of this paper, please visit http://youtu.be/2fbKtDAOqQ0.

1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers and $k \geq 2$ be a fixed integer. For any set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let

$$r_k(A,n) = \sharp\{(a_1, \ldots, a_k) \in A^k:\ a_1 + \cdots + a_k = n\}.$$

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We call $A \subseteq \mathbb{N}$ an asymptotic basis of order k if there is $n_0 = n_0(A)$ such that $r_k(A, n) \geq 1$ for each positive integer $n \geq n_0$. In particular, we call $A \subseteq \mathbb{N}$ a basis of order k if $r_k(A, n) \geq 1$ for all $n \geq 0$.

The well-known Erdős–Turán conjecture [4] states that if A is an asymptotic basis of order 2, then $r_2(A, n)$ cannot be bounded. This harmlessly looking conjecture has attracted many mathematics since 1941. But, to our regret, not much is known about Erdős–Turán conjecture itself. Erdős–Turán conjecture seems to be extremely difficult. In 1990, Ruzsa [7] found a basis A of $\mathbb{N}$ for which $r_2(A, n)$ is bounded in the square mean. That is, he constructed a basis $A \subseteq \mathbb{N}$ satisfying $\sum_{n \leq N} r_2^2(A, n) = O(N)$. Based on the method of Ruzsa, Tang [8] gave a quantitative version of Ruzsa's theorem. Recently, Chen and Yang [3] gave a new better bound in Ruzsa's theorem. They announced that they can find a basis of A of $\mathbb{N}$ such that $\sum_{n \leq N} r_2^2(A, n) \leq 2920N$ for all $N \geq 1$. In 2003, Grekos et al. [5] proved that if A is a basis of order 2, then $\limsup_{n\to\infty} r_2(A, n) \geq 6$. In 2005, Borwein et al. [1] improved 6 to 8. In 2012, Chen [2] proved that there is a basis A of order 2 such that the set of n with $r_2(A, n) = 2$ has density one. Recently, Yang [9] extended Chen's theorem to basis of order k, he proved the following result.

Theorem A. *For any integer $k \geq 2$, there is a basis $A_k \subseteq \mathbb{N}$ of order k such that the set of n with $r_k(A_k, n) = k!$ has density one.*

In this paper, we extend Chen and Yang's results as follows.

Theorem 1. *Let $k \geq 2$ be a fixed integer. Then for any positive integer u, there is a set $A \subseteq \mathbb{N}$ such that $r_k(A, n) \geq 1$ for all $n \geq 0$ and the set of n with $r_k(A, n) = k!u$ has density one.*

2. Proofs

Lemma 1. *(See [6, Theorem 143].) Almost all positive integers, when expressed in any scale, contain a given possible sequence of digits.*

Proof of Theorem 1. By Theorem A, we only need to consider the case $u \geq 2$. Now we suppose that $u \geq 2$.

For $j = 0, 1, \ldots, k-1$, let

$$A_j = \left\{ \sum_{i=0}^{\infty} \varepsilon_i k^{ki+j} \colon \varepsilon_i \in \{0, 1, \ldots, k-1\} \right\} = \left\{0 = a_1^{(j)} < a_2^{(j)} < \cdots\right\},$$

where in each sum there are only finitely many $\varepsilon_i \neq 0$.

Let

$$H = \left\{0 = a_1^{(0)} < \cdots < a_u^{(0)}\right\}.$$