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# Indices of inseparability in towers of field extensions

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ABSTRACT

Let $K$ be a local field whose residue field has characteristic $p$ and let $L/K$ be a finite separable totally ramified extension of degree $n = ap^{\nu}$. The indices of inseparability $i_0, i_1, \dots, i_{\nu}$ of $L/K$ were defined by Fried in the case $\mathrm{char}(K) = p$ and by Heiermann in the case $\mathrm{char}(K) = 0$; they give a refinement of the usual ramification data for $L/K$. The indices of inseparability can be used to construct "generalized Hasse–Herbrand functions" $\phi^j_{L/K}$ for $0 \le j \le \nu$. In this paper we give an interpretation of the values $\phi^j_{L/K}(c)$ for nonnegative integers $c$. We use this interpretation to study the behavior of generalized Hasse–Herbrand functions in towers of field extensions.



## 1. Introduction

Let $K$ be a local field whose residue field $\overline{K}$ is a perfect field of characteristic $p$, and let $K^{sep}$ be a separable closure of $K$. Let $L/K$ be a finite totally ramified subextension of $K^{sep}/K$. The *indices of inseparability* of $L/K$ were defined by Fried [2] in the case $\mathrm{char}(K) = p$, and by Heiermann [5] in the case $\mathrm{char}(K) = 0$. The indices of inseparability of $L/K$ determine the ramification data of $L/K$ (as defined for instance in Chapter IV of [7]), but the ramification data does not always determine the indices of inseparability.

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Therefore the indices of inseparability of $L/K$ may be viewed as a refinement of the usual ramification data of $L/K$.

Let $\pi_K$, $\pi_L$ be uniformizers for $K$, $L$. The most natural definition of the ramification data of $L/K$ is based on the valuations of $\sigma(\pi_L) - \pi_L$ for $K$-embeddings $\sigma : L \to K^{sep}$; this is the approach taken in Serre's book [7]. The ramification data can also be defined in terms of the relation between the norm map $\mathrm{N}_{L/K}$ and the filtrations of the unit groups of $L$ and $K$, as in Fesenko–Vostokov [1]. This approach can be used to derive the well-known relation between higher ramification theory and class field theory. Finally, the ramification data can be computed by expressing $\pi_K$ as a power series in $\pi_L$ with coefficients in the set $R$ of Teichmüller representatives for $\overline{K}$. This third approach, which is used by Fried and Heiermann, makes clear the connection between ramification data and the indices of inseparability.

Heiermann [5] defined "generalized Hasse–Herbrand functions" $\phi^j_{L/K}$ for $0 \le j \le \nu$. In Section 2 we give an interpretation of the values $\phi^j_{L/K}(c)$ of these functions at non-negative integers $c$. This leads to an alternative definition of the indices of inseparability which is closely related to the third method for defining the ramification data. In Section 3 we consider a tower of finite totally ramified separable extensions $M/L/K$. We use our interpretation of the values $\phi^j_{L/K}(c)$ to study the relations between the generalized Hasse–Herbrand functions of $L/K$, $M/L$, and $M/K$.

**Notation**

$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$
$v_p = p$-adic valuation on $\mathbb{Z}$
$K =$ local field with perfect residue field $\overline{K}$ of characteristic $p > 0$
$K^{sep} =$ separable closure of $K$
$v_K =$ valuation on $K^{sep}$ normalized so that $v_K(K^\times) = \mathbb{Z}$
$\mathcal{O}_K = \{\alpha \in K : v_K(\alpha) \ge 0\} =$ ring of integers of $K$
$\pi_K =$ uniformizer for $K$
$\mathcal{M}_K = \pi_K \mathcal{O}_K =$ maximal ideal of $\mathcal{O}_K$
$R =$ set of Teichmüller representatives for $\overline{K}$
$L/K =$ finite totally ramified subextension of $K^{sep}/K$ of degree $n > 1$, with $v_p(n) = \nu$
$M/L =$ finite totally ramified subextension of $K^{sep}/L$ of degree $m > 1$, with $v_p(m) = \mu$
$v_K$, $\mathcal{O}_K$, $\pi_K$, and $\mathcal{M}_K$ have natural analogs for $L$ and $M$

## 2. Generalized Hasse–Herbrand functions

We begin by recalling the definition of the indices of inseparability $i_j$ $(0 \le j \le \nu)$ for a nontrivial totally ramified separable extension $L/K$ of degree $n = ap^\nu$, as formulated by Heiermann [5]. Let $R \subset \mathcal{O}_K$ be the set of Teichmüller representatives for $\overline{K}$. Then there is a unique series $\hat{\mathcal{F}}(X) = \sum_{h=0}^{\infty} a_h X^{h+n}$ with coefficients in $R$ such that $\pi_K = \hat{\mathcal{F}}(\pi_L)$. For $0 \le j \le \nu$ set