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Proof of a conjecture of Wong concerning octahedral Galois representations of prime power conductor

Kevin Childers, Darrin Doud *

Department of Mathematics, Brigham Young University, Provo, UT 84602, United States

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ABSTRACT

We prove a conjecture of Siman Wong concerning octahedral Galois representations of prime power conductor.

1. Introduction

Let $\bar{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$, and write $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In this paper a Galois representation is defined as a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{C})$. It is well known that such a representation must have finite image. In fact, if $\pi : \mathrm{GL}(2, \mathbb{C}) \to \mathrm{PGL}(2, \mathbb{C})$ is the standard quotient map, $\tilde{\rho} = \pi \circ \rho$ has an image that is either cyclic or isomorphic to a dihedral group, A_4, S_4, or A_5. A Galois representation is said to be odd if it maps complex conjugation to a nonscalar matrix, and is said to be even otherwise.

* Corresponding author.
E-mail addresses: kevinrchilders@gmail.com (K. Childers), doud@math.byu.edu (D. Doud).

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Given a projective representation $\tilde{\rho} : G_{\mathbb{Q}} \to \mathrm{PGL}(2, \mathbb{C})$, a lift of $\tilde{\rho}$ will be any Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{C})$ such that $\tilde{\rho} = \pi \circ \rho$.

A Galois representation is ramified at p if the image of an inertia group at p under ρ is nontrivial. The conductor of a Galois representation is a product of powers of primes at which it is ramified. For tamely ramified primes, the exponent of p in this product is easily described: if we let $G_{\mathbb{Q}}$ act on $\mathbb{C}^2$ via ρ, the exponent of p in the conductor is the codimension of the fixed space of inertia at p [3, p. 527].

Given a projective representation $\tilde{\rho} : G_{\mathbb{Q}} \to \mathrm{PGL}(2, \mathbb{C})$, Serre [4, §6.2] defines the conductor of $\tilde{\rho}$ as a product over all primes p of local conductors. For each prime p, let $\tilde{\rho}_p = \tilde{\rho}|_{D_p}$ be the restriction of $\tilde{\rho}$ to a decomposition group at p. The local conductor at p is the minimum conductor of all lifts to $\mathrm{GL}(2, \mathbb{C})$ of $\tilde{\rho}_p$. Each of these local conductors is a power of p; for unramified primes the exponent is 0, and for tamely ramified p the exponent is 1 if the image of $\tilde{\rho}_p$ is cyclic and 2 otherwise [4, §6.3].

Because our Galois representations have domain $G_{\mathbb{Q}}$, we may also describe the conductor of a projective representation $\tilde{\rho}$ as the minimum of the conductors of all the lifts of $\tilde{\rho}$ [4, §6.2].

Serre [4] classified all odd projective Galois representations of prime conductor, and Vignéras [6] classified all even projective representations of prime conductor. More recently, Siman Wong [7] studied octahedral representations (representations with projective image isomorphic to S_4) of prime power conductor and made the following conjecture about these representations:

Theorem 1.1. *(See [7, Conjecture 2].) Let $K_4/\mathbb{Q}$ be an S_4-quartic field such that $|d_{K_4}|$ is a power of a prime $p > 3$. Let $K_3/\mathbb{Q}$ be a cubic subfield of the Galois closure of $K_4/\mathbb{Q}$. Denote by $\tilde{\rho}$ the projective 2-dimensional Artin representation associated to $K_4/\mathbb{Q}$.*

1. *Suppose $K_3/\mathbb{Q}$ is totally real. If $\tilde{\rho}$ has conductor p^2, then $v_p(d_{K_4}) = 1$.*
2. *Suppose $K_3/\mathbb{Q}$ is not totally real. If $\tilde{\rho}$ has conductor p^2 then $v_p(d_{K_4}) = 3$, otherwise $v_p(d_{K_4}) = 1$.*

In this paper, we apply techniques of Serre to prove Wong's conjecture (see Section 3).

2. Background

For a number field K, we will denote the discriminant of K by d_K. We note that Stickelberger's criterion [1, p. 67] implies that for any number field K, d_K is congruent to 0 or 1 modulo 4. All discriminants that we consider will be odd, so we will always have $d_K \equiv 1 \pmod 4$.

Throughout this paper, $K_4/\mathbb{Q}$ will denote a field extension of degree 4 with Galois group S_4 and discriminant a power of a prime $p > 3$. We will denote by $K_3/\mathbb{Q}$ a cubic subextension of the splitting field of $K_4/\mathbb{Q}$.

Given $K_4/\mathbb{Q}$, there will be an associated projective Galois representation $\tilde{\rho} : G_{\mathbb{Q}} \to \mathrm{PGL}(2, \mathbb{C})$ with image isomorphic to S_4. Since K_4 is ramified only at p, $\tilde{\rho}$ will be ramified