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Families of curves with Galois action and their L-functions

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ABSTRACT

We generalise results of Chris Hall on the L-function of curves E over characteristic p function fields K, by using equivariant L-functions and cohomologically trivial modules. In fact, K will be the rational function field over a fixed finite field most of the time. The curves which we can treat are superelliptic curves which come as Galois covers of prime degree of the projective line over K. We are thus able to determine the degree of the L-function (which is a polynomial in our situation), and sometimes we get upper bounds on the analytic rank.

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0. Introduction

In the paper [Ha], Chris Hall looked at the Hasse–Weil L-function $L(T, E/K)$ of certain elliptic curves over the function field K of a curve C over a fixed finite field $\mathbb{F}_q$, assuming that the j-invariant of E is not constant. In other words, he considered a nontrivial family of curves indexed with the points of C, where almost all fibres are elliptic curves, and a finite number of fibres may be singular. For the computation of the L-function it is practical to fix a Weierstraß model $\mathcal{E} \to C$. The special case $C = \mathbb{P}^1$ is of central importance in Hall's paper, and we shall essentially restrict ourselves to this situation.

One of the main results of Hall concerns a family of elliptic curves over C (with finitely many singular fibres) given by a simple Legendre equation. Cleverly using the presence of 2-torsion points in the fibres, he proved a congruence modulo 2 for the L-function, which is in fact a polynomial. This gives the degree of $L(T, E/K)$ at once, some information about the analytic rank of E/K, and in some cases actually a verification of the rank part of the Birch–Swinnerton-Dyer conjecture.

Hall's arguments are basically very explicit. It is our goal here to demonstrate that these results are underpinned by Galois theory and cohomology, and that they can be generalised to superelliptic curves that are cyclic covers of the projective line $\mathbb{P}^1_K$, if one is willing to accept some technical hypotheses. In this approach cohomologically trivial modules play a big role. Our philosophy can be summarised in two statements. (1) Equivariant L-functions that are constructed using cohomologically trivial modules have better integrality properties over the group ring than L-functions which are constructed via the characters of the group and then assembled. (2) Integrality of an object over the group ring translates into congruences between the images of that object under the characters of the group. (We are not claiming that either of these statements is at all new.) This underpinning by module-theoretic concepts is one main difference with respect to Hall's paper. The other major difference is the fact that for more general curves there is no fast and simple analog of the minimal Weierstraß model of an elliptic curve, so new technicalities arise; we will say a little more on this near the end of the introduction.

Our main general result may be stated as follows, omitting a few details now. We consider a superelliptic curve E/K over $K = \mathbb{F}_q(t)$ which is not constant (that is, not induced from a curve over $\mathbb{F}_q$) and comes as a cyclic cover of the projective line $\mathbb{P}^1_K$, with $K = \mathbb{F}_q(t)$ the function field of the projective line over $\mathbb{F}_q$. This projective line with coordinate t is written C, so $K = \mathbb{F}_q(C) = \mathbb{F}_q(t)$. We continue to denote the curve over K by E, even though it need not be an elliptic curve; its genus g_E will usually be greater than 1. The curve E is given by an affine equation

$$y^r = (x - f_1(t)) \cdots (x - f_d(t)),$$