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On a conjecture of de Koninck ☆

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A B S T R A C T

For a positive integer n, let $\sigma(n)$ and $\gamma(n)$ denote the sum of divisors and the product of distinct prime divisors of n, respectively. It is known that, if $\sigma(n) = \gamma(n)^2$, then at most two exponents of odd primes are equal to 1 in the prime factorization of n. In this paper, we prove that, if $\sigma(n) = \gamma(n)^2$ and only one exponent is equal to 1 in the prime factorization of n, then (1) n is divisible by 3; (2) n is divisible by the fourth powers of at least two odd primes; (3) at least two exponents of odd primes are equal to 2. We also prove that, if $\sigma(n) = \gamma(n)^2$, then at least half of the exponents α of the primes have the property that the numbers $\alpha + 1$ must be either primes or prime squares.

1. Introduction

For a positive integer n, let $\sigma(n)$ and $\gamma(n)$ denote the sum of divisors and the product of distinct prime divisors of n, respectively. De Koninck (see Guy [5, Section B11]) conjectured that the equation

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$$\sigma(n) = \gamma(n)^2 \tag{1.1}$$

has only solutions $n = 1$ and $n = 1728$. Let $\mathcal{K}$ denote the set of all solutions n to (1.1).

Broughan, de Koninck, Kátai and Luca [3] proved that Eq. (1.1) with $\omega(n) \le 4$ has only solutions $n = 1$ and $n = 1728$, and if $n > 1$ and $n \in \mathcal{K}$, then the prime factorization of n has the form

$$n = 2^\alpha p \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{1.2}$$

where $\alpha \ge 1$, α_i $(2 \le i \le s)$ are even and either $p \equiv p_1 \equiv \alpha_1 \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 8$ and α_1 is even. This is equivalent to

Theorem A. *If $n > 1$ and $n \in \mathcal{K}$, then the prime factorization of n has the form either*

$$n = 2^\alpha pq \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{1.3}$$

where $\alpha \ge 1$, α_i $(1 \le i \le s)$ are even and $p \equiv q \equiv 1 \pmod 4$, or

$$n = 2^\alpha p \prod_{i=1}^{s} p_i^{\alpha_i}, \tag{1.4}$$

where $\alpha \ge 1$, α_i $(2 \le i \le s)$ are even and either $p \equiv p_1 \equiv \alpha_1 \equiv 1 \pmod 4$, $\alpha_1 \ge 5$ or $p \equiv 3 \pmod 8$ and α_1 is even.

Recently, Broughan, Delbourgo and Zhou [4] proved the following result:

Theorem B. *(See [4, Theorem 1].) If $n \in \mathcal{K}$ and $n > 1$, then n is divisible by the fourth power of an odd prime.*

In this paper, the following results are proved.

Theorem 1.1. *If $n > 1$ and $n \in \mathcal{K}$ with the form (1.4), then $3 \mid n$.*

Theorem 1.2. *If $n > 1$, $n \ne 1782 = 2 \cdot 3^4 \cdot 11$ and $n \in \mathcal{K}$ with the form (1.4), then n is divisible by the fourth powers of at least two odd primes.*

Theorem 1.3. *If $n > 1$, $n \ne 1782 = 2 \cdot 3^4 \cdot 11$ and $n \in \mathcal{K}$ with the form (1.4), then, $p \ge 1571$ and at most two of $p_1, \dots, p_s$ are larger than p. Moreover, if $p \le 10p_i^2$, then $\alpha_i = 2$.*

Remark 1. From the proof, it is easy to see that 1571 and 10 in Theorem 1.3 can be improved. We do not pursue these bounds.