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Classification of algebraic function fields with class number one

Pietro Mercuri, Claudio Stirpe *

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ABSTRACT

In this paper we prove that there are exactly eight function fields, up to isomorphism, over finite fields with class number one and positive genus. This classification was already suggested, although not completely proved, in a previous work about this topic (see Stirpe [7]).

1. Introduction

The problem of the determination of algebraic function fields over finite fields with class number one was already treated in the paper of Leitzel, Madan and Queen [2]. But, in Stirpe [7], one more example is given, showing that the previous classification is incomplete. In this paper we give the full list of function fields with class number one and positive genus hence their classification is now complete.

The list of all function fields with class number one and positive genus is given in the following theorem.

* Corresponding author.
E-mail addresses: mercuri.ptr@gmail.com (P. Mercuri), clast@inwind.it (C. Stirpe).

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Theorem 1.1. *Let K be a function field over the finite field $\mathbb{F}_q$ of genus $g > 0$ with class number one. Then K is isomorphic to a function field $\mathbb{F}_q(x, y)$ defined by one of the following equations:*

(i) $y^2 + y + x^3 + x + 1 = 0$, *with* $g = 1$ *and* $q = 2$;
(ii) $y^2 + y + x^5 + x^3 + 1 = 0$, *with* $g = 2$ *and* $q = 2$;
(iii) $y^2 + y + (x^3 + x^2 + 1)/(x^3 + x + 1) = 0$, *with* $g = 2$ *and* $q = 2$;
(iv) $y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + x^4 + x + 1 = 0$, *with* $g = 3$ *and* $q = 2$;
(v) $y^4 + (x^3 + x + 1)y + x^4 + x + 1 = 0$, *with* $g = 3$ *and* $q = 2$;
(vi) $y^2 + 2x^3 + x + 1 = 0$, *with* $g = 1$ *and* $q = 3$;
(vii) $y^2 + y - x^3 + \alpha = 0$, *with* $g = 1$ *and* $q = 4$, *where* α *is a generator of the multiplicative group* $\mathbb{F}_4^*$;
(viii) $y^5 + y^3 + y^2(x^3 + x^2 + x) + y(x^7 + x^5 + x^4 + x^3 + x)/(x^4 + x + 1) + (x^{13} + x^{12} + x^8 + x^6 + x^2 + x + 1)/(x^4 + x + 1)^2 = 0$, *with* $g = 4$ *and* $q = 2$.

Function fields (i–vii) are already given in [2] but in that paper the authors wrongly claimed to have ruled out the only other possibility of curves of genus 4 over $\mathbb{F}_2$. Madan and Queen also showed in [3] that any other example of function field with class number one and positive genus should be a function field of genus 4 with exactly one place of degree 4 and without places of smaller degree. A function field with this properties was later found in [7] but the author did not prove that such example is unique up to isomorphism. This function field is listed in (viii) and we prove uniqueness in Section 3 giving a definitive proof of Theorem 1.1. The proof is elementary and requires only basic facts about function fields and Class Field Theory. The necessary background is given in the next section.

In the fourth section we show another proof that is a complete version of the argument in [2]. The interested reader can also find a simplified argument in Qibin Shen and Shuhui Shi [4] which was done independently around the same time.

2. Background and definitions

Given two extensions of function field K and L of $\mathbb{F}_q(x)$, we assume that they are embedded in the same algebraic closure of $\mathbb{F}_q(x)$.

Let K/F be a separable extension of function fields. Let n be the degree $[K : F]$ and let g_K and g_F be the genus of K and F, respectively. We assume that the constant fields of F and K are the same. Then the genera of F and K are related by the Hurwitz Genus Formula:

$$2g_K - 2 = n(2g_F - 2) + \deg \mathrm{Diff}(K/F), \tag{2.1}$$

where $\mathrm{Diff}(K/F) = \sum_P d_P P$ is the different of K/F and the sum runs over the ramified places of K. The reader can see Stichtenoth [5, Chapter 3] for definitions and main